ON RECURRENCE AND ERGODICITY FOR GEODESIC FLOWS ON NONCOMPACT PERIODIC POLYGONAL SURFACES

JEAN-PIERRE CONZE AND EUGENE GUTKIN

Abstract. We study the recurrence and ergodicity for the billiard on noncompact polygonal surfaces with a free, cocompact action of $\mathbb{Z}$ or $\mathbb{Z}^2$. In the $\mathbb{Z}$-periodic case, we establish criteria for recurrence. In the more difficult $\mathbb{Z}^2$-periodic case, we establish some general results. For a particular family of $\mathbb{Z}^2$-periodic polygonal surfaces, known in the physics literature as the wind-tree model, assuming certain restrictions of geometric nature, we obtain the ergodic decomposition of directional billiard dynamics for a dense, countable set of directions. This is a consequence of our results on the ergodicity of $\mathbb{Z}^2$-valued cocycles over irrational rotations.

Contents

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Introduction

Beginning with Boltzmann's *ergodic hypothesis*, mathematicians have been investigating the ergodicity of dynamical systems of physical origin. Among them are the geodesic flows on riemannian configuration spaces describing mathematically the physical models at hand. If the configuration space has a boundary, we arrive at a billiard.

It is notoriously difficult to study the ergodicity of these dynamical systems, in particular the famous Boltzmann-Sinai model. On the contrary, the conservativeness of a dynamical system of this kind is guaranteed by the Poincaré recurrence theorem, provided its phase space has finite volume. This holds, for instance, if the configuration space is compact.

The situation changes drastically if the configuration space has infinite volume. This happens, in particular, if the space is invariant under a free action of an infinite group, say $\mathbb{Z}^d$. Not only the ergodicity, but even the conservativeness of these dynamical systems is a challenging question; it is open in many relevant examples.

Some physical models correspond to the geodesic flows on polygonal surfaces invariant under free actions of infinite groups [13]. We will speak of *periodic polygonal surfaces* or *G-periodic polygonal surfaces*, where G is the group in question. For instance, the space of the classical wind-tree model in statistical physics [16] is a $\mathbb{Z}^2$-periodic polygonal surface.

In this work we study the recurrence and ergodicity for geodesic flows on noncompact polygons and noncompact polygonal surfaces. For reader's convenience, we will briefly survey the relevant material in the compact case. We refer to [10, 11] for details. Let P be a compact polygonal surface, e. g., a polygon. If P is rational, the study of the geodesic flow on P is equivalent to the study of the geodesic flow on a compact translation surface, say S. That flow decomposes as a one-parameter family of directional translation flows, say $T_\theta^t : S \to S, 0 \leq \theta \leq 2\pi$. The celebrated result in this subject says that for Lebesgue almost all directions θ the flows T_θ^t are (uniquely) ergodic [19]. This theorem has far reaching applications to the billiard in irrational polygons [19, 23]. See [11, 12] for details.

Let now S be a noncompact translation surface. Let $T_\theta^t : S \to S$, $0 \le \theta \le 2\pi$, be the one-parameter family of directional translation flows [13]. For the purposes of this discussion we assume that S is a periodic translation surface. The following question naturally arises: Is there an analog of the unique ergodicity theorem in [19] for noncompact, periodic translation surfaces? This question is mainly open. In fact, it is not known whether the flows $T_\theta^t : S \to S$ are conservative for typical directions θ. The examples from our sections 2, 3 and 4 might be useful to formulate conjectures about the flows $T_\theta^t : S \to S$ for noncompact translation surfaces.

We will now informally describe some of our results. Let $\tilde{P}$ be a $\mathbb{Z}$-periodic polygonal surface with a boundary. Let $P = \tilde{P}/\mathbb{Z}$ be the compact quotient. Suppose that the billiard flow on P is ergodic. Then the billiard flow on $\tilde{P}$ is conservative. See Theorem 1.

With any polygon O inside the unit square we associate the $\mathbb{Z}$-periodic strip $\tilde{P}_O$ with polygonal obstacles. Then for a dense G_δ-set of obstacles, the billiard flow on $\tilde{P}_O$ is conservative. See Theorem 3. Let O be an *irrational polygon*. Suppose that its angles admit a superexponentially fast approximation by numbers in $\pi\mathbb{Q}$. Then the geodesic flow on $\tilde{P}_O$ is conservative. See Theorem 4.

Let $\tilde{P} = \tilde{P}(a, b)$ be the *rectangular Lorenz gas* obtained by deleting from $\mathbb{R}^2$ the $\mathbb{Z}^2$-periodic family of $a \times b$ rectangles. This corresponds to the *wind-tree model* [16]. Let $p, q \in \mathbb{N}$ be relatively prime; denote by $\tilde{T}(a, b; p, q)$ the billiard flow in the direction $\arctan(q/p)$. We say that the *obstacles are small* if $qa + pb \le 1$. Assuming the small obstacles condition, we analyze the flow $\tilde{T}(a, b; p, q)$. Let $\tilde{T}_{\text{cons}}(a, b; p, q)$ and $\tilde{T}_{\text{diss}}(a, b; p, q)$ be the conservative and the dissipative parts of $\tilde{T}(a, b; p, q)$ respectively. We show that $\tilde{T}_{\text{diss}}(a, b; p, q)$ is trivial iff $qa + pb = 1$. Assume now that a/b is irrational. Then we obtain an ergodic decomposition of $\tilde{T}_{\text{cons}}(a, b; p, q)$. The $2pq$ ergodic components are isomorphic; they have a simple geometric meaning. Thus, $\tilde{T}_{\text{cons}}(a, b; p, q)$ is a finite multiple of an ergodic flow. See Theorem 5, Theorem 6, and Proposition 10.

For instance, the conservative part of the wind-tree billiard flow in direction $\pi/4$ is the flow $\tilde{T}_{\text{cons}}(a, b; 1, 1)$; it has two ergodic components. Figure 1 shows a typical orbit of this flow. It encounters only a half of the set of rectangular obstacles. Loosely speaking, the orbit skips every other obstacle. The skipped obstacles are visited by a typical orbit from the other ergodic component of the flow.

This is a special case of the general situation, as we explain in Theorem 5 and Theorem 6.

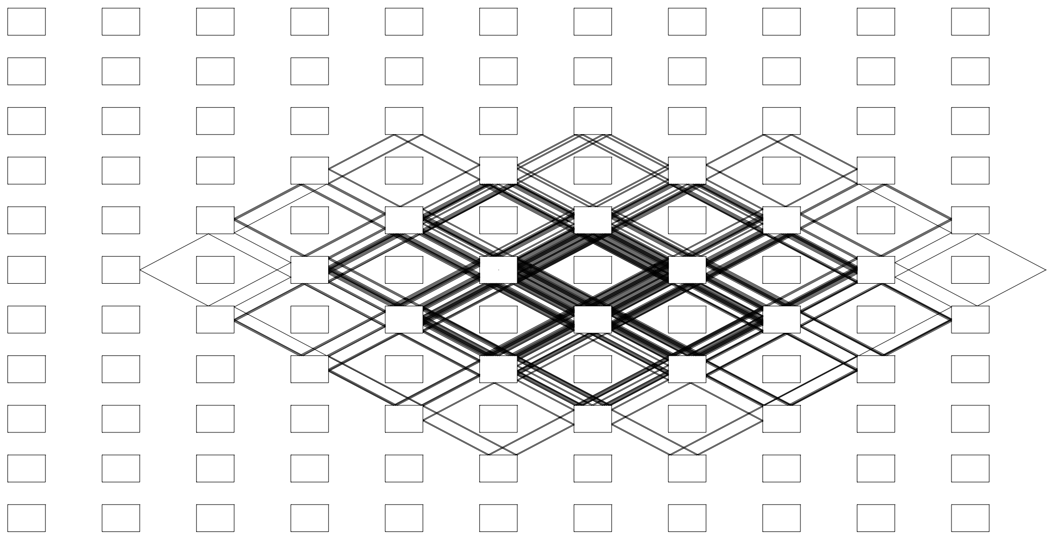

FIGURE 1. *An orbit for the flow $\tilde{T}_{cons}(a, b; 1, 1)$.*

We will now comment on our methods. The billiard flows are suspensions of billiard maps. Exploiting the periodicity under a group G, we identify a billiard map of this kind with a skew product over an interval exchange with the fiber G. In our setting, we obtain skew products with the fibers $\mathbb{Z}$ and $\mathbb{Z}^2$. Thus, we reduce the questions concerning the ergodicity of the billiard on noncompact polygonal surfaces to the ergodicity of particular $\mathbb{Z}^d$-valued cocycles over interval exchanges.

This work is not concerned with the subject of ergodicity for cocycles over general interval exchanges. We establish the ergodicity of a class of $\mathbb{Z}^2$-valued cocycles over irrational rotations. See Theorems 7, 8, and 9 in section 5. Theorem 7 states that a cocycle of this kind is ergodic if the continued fraction decomposition of the rotation number satisfies certain genericity assumptions which hold for almost all numbers. Theorem 8 and Theorem 9 strengthen Theorem 7 by removing these assumptions.

We apply these results in section 4 to obtain the ergodic decompositions of directional flows in the wind-tree model. Let T^t be one of the $2pq$ geometric components of $\tilde{T}_{\text{cons}}(a, b; p, q)$. The Poincaré map for T^t is a skew product with fibre $\mathbb{Z}^2$ over a circle rotation; this rotation is irrational iff a/b is irrational. The corresponding $\mathbb{Z}^2$-valued cocycle

belongs to the class of cocycles studied in section 5. Theorem 9 implies the ergodicity of T^t.

We will now outline the structure of our exposition. Section 1 contains the information to be used in the body of the paper. In section 1.1 we review the material on skew products, cocycles, recurrence and transience. In section 1.2 we establish the framework of polygonal surfaces. In section 1.3 we recall the basic facts about the billiard flow and the billiard map.

Section 2 is about the $\mathbb{Z}$-periodic case. This framework is naturally divided into two extreme situations: the generic case and the rational case. In the former situation we show the conservativeness and some ergodic properties for the generic $\mathbb{Z}$-periodic polygonal surface; in the latter we establish these properties for the directional flows in almost all directions.

Section 3 and section 4 are devoted to the billiard on $\mathbb{Z}^2$-periodic polygonal surfaces. In section 3 we establish some ergodic properties for arbitrary $\mathbb{Z}^2$-periodic polygonal surfaces. In section 4 we consider a particular family of such surfaces: The rectangular Lorenz gas or the wind-tree model of the physics literature. Applying the results of section 5, we obtain the ergodic decomposition of directional billiard flows for a dense, countable set of directions, under restrictions of geometric nature, namely the smallness of obstacles condition. Section 5 is a study of ergodicity for a class of cocycles over irrational rotations. The results, besides being of interest on their own, are instrumental for the material in section 4.

1. The setting and preliminaries

For convenience of the reader, we recall the basic material about recurrence [1, 7, 22]. We will consider two kinds of *dynamical systems*: transformations and flows. In the former case, we have the standard Borel space $(X, \mathcal{A})$ endowed with a possibly infinite measure ν, and a transformation $\tau : (X, \mathcal{A}) \to (X, \mathcal{A})$ preserving ν. For simplicity, we will assume that τ is invertible. The setting for flows is analogous [7]. We will use the notation (X, τ, ν) (resp. (Y, T^t, μ)) for transformations (resp. flows).

The dynamical system (X, τ, ν) is *recurrent* or *conservative* if for every measurable set $B \subset X$ and for ν-a.e. point $x \in B$ there is $n = n(x) > 0$ such that $\tau^n x \in B$. Recurrence for flows (Y, T^t, μ) is defined analogously. A dynamical system uniquely decomposes as a disjoint union of the *conservative* part and the *dissipative part.* For simplicity, we describe this decomposition only for a transformation, (X, τ, ν).

The conservative, dissipative subsets $C, D \subset X$ are measurable and τ-invariant. If $\nu(C) > 0$ then $(C, \tau, \nu) \subset (X, \tau, \nu)$ is recurrent. Suppose that $\nu(D) > 0$. Then there is a measurable set $A \subset D$ such that $D = \cup_{n \in \mathbb{Z}} \tau^n A$; moreover, $\tau^p A \cap \tau^q A = \emptyset$ for $p \neq q$.

If $\nu(X) = \infty$, this decomposition is, in general, nontrivial. We will use the following observation. Let (Y, T^t, μ) be a flow, let $X \subset Y$ be a *cross-section*, and let (X, τ, ν) be the induced transformation. Then the transformation (X, τ, ν) is recurrent iff the flow (Y, T^t, μ) is recurrent.

The geometric spaces that we work with in the body of the paper are differentiable manifolds, possibly with boundary and corners. The transformations and flows are piecewise differentiable.

1.1. **Ergodic theory for skew products.**

We will represent our dynamical systems as skew products over dynamical systems with finite invariant measures. Their fibers will be infinite abelian groups.

In this section we recall the relevant material on G-valued cocycles, where G is an infinite abelian group; we will write the group operation additively. We restrict the discussion mostly to the groups $G = \mathbb{R}^m \times \mathbb{Z}^n$. We denote by Leb_G a Haar measure on G, suppressing the subscript if the group is clear from the context.

Definition 1. *Let (X, τ, ν) be a dynamical system, and let $\varphi : X \to G$ be a measurable function. It determines a* cocycle $\varphi(n, x)$*, also denoted by $\varphi_n(x)$ or simply (φ_n), as follows. We set $\varphi(0, x) = 0$. For $n \neq 0$ we set*

$$\varphi_n(x) = \sum_{j=0}^{n-1} \varphi(\tau^j x), \text{ if } n > 0; \ \ \varphi_n(x) = -\sum_{j=n}^{-1} \varphi(\tau^j x), \text{ if } n < 0. \tag{1}$$

Thus, $\varphi(n, x)$ are the ergodic (or Birkhoff) sums of φ with respect to the transformation (X, τ, ν). The cocycle (φ_n) can be viewed as the random walk on the group G driven by the dynamical system (X, τ, ν). Set

$$\tilde{\tau}(x, g) = (\tau x, g + \varphi(x)). \tag{2}$$

Then $\tilde{\tau}$, or τ_φ to emphasize the dependence on φ, is a transformation of $\tilde{X} = X \times G$ preserving the product measure $\tilde{\nu} = \nu \times \mathrm{Leb}$. The dynamical system $(\tilde{X}, \tilde{\tau}, \tilde{\nu})$ (or $(\tilde{X}, \tau_\varphi, \tilde{\nu})$) is the *skew product* over (X, τ, ν) with the *displacement function* φ. Equation (1) corresponds to the iterates of $\tilde{\tau}$. Namely, for $n \in \mathbb{Z}$ we have

$$\tilde{\tau}^n(x, g) = (\tau^n(x), g + \varphi_n(x)). \tag{3}$$

Definition 2. *Let (X, τ, ν) be a dynamical system with $\nu(X) < \infty$. Let $\varphi : X \to G$ be a measurable function. The cocycle (φ_n) is transient at $x \in X$ if $\varphi_n(x) \to \infty$; otherwise, the cocycle is* recurrent at x.[1] *The cocycle is recurrent (resp. transient) if it is recurrent (resp. transient) at a.e. $x \in X$.*

We point out a subtlety in Definition 2. Let α_n, β_n be $\mathbb{Z}$-valued cocycles over (X, τ, ν). Their direct sum $\varphi_n = (\alpha_n, \beta_n)$ is a $\mathbb{Z}^2$-valued cocycle. The recurrence of α_n, β_n does not necessarily imply that φ_n is recurrent. See, e. g., [4] for an example.

The sets of transient and recurrent points for a cocycle are measurable and invariant. Hence, any cocycle φ_n over an ergodic (X, τ, ν) is either recurrent or transient. Let (X, τ, ν) be arbitrary, let (φ_n) be a cocycle, and let $R \subset X$ be the set of recurrent points for φ_n. Suppose that $\nu(R) > 0$, and let ν_R be the restriction of ν to R; set $\tilde{\nu}_R = \nu_R \times \text{Leb}$, $\tilde{R} = R \times G$. Then the skew product $(\tilde{R}, \tilde{\tau}, \tilde{\nu}_R)$ is a conservative dynamical system [22]. Assume, moreover, that X is a separable metric space and that $\tau : X \to X$ is compatible with the topological structure.[2] Then for a.e. $x \in R$ there is an infinite sequence $n_k = n_k(x)$ such that $\tau^{n_k} x \to x$ and $\varphi(n_k, x) \to 0$ [22]. Therefore, for almost every point in R the sequence $(\varphi_k(x))_{k \geq 0}$ visits arbitrarily close to $0 \in G$. If $G = \mathbb{Z}^d$, then we have $\varphi_k(x) = 0$ infinitely many times. Suppose now that $(\tilde{X}, \tilde{\tau}, \tilde{\nu})$ comes from a billiard model. Then R is the conservative part of the billiard phase space. The billiard ball emanating from a point in R almost surely returns infinitely often to the obstacle from which it started, and arbitrarily close to the point of departure.

We refer the reader to [22] for criteria of recurrence for cocycles. See also [20], [2]. The following lemma from [4] gives a simple sufficient condition for recurrence of $\mathbb{R}^d$-valued cocycles.

Lemma 1. *Let (X, τ, ν) be a dynamical system with a finite measure, let $\varphi : X \to \mathbb{R}^d$ be a measurable function, and let $\varphi(n, x)$ be the corresponding cocycle.*

Let $|\ |$ be a norm on $\mathbb{R}^d$. Suppose that there exists a strictly increasing sequence of integers k_n and a sequence of nonnegative functions $\delta_n(x)$ converging to 0 for almost every x such that

$$\lim_n \nu(\{x : |\varphi(k_n, x)| \geq \delta_n(x) n^{\frac{1}{d}}\}) = 0. \tag{4}$$

Then the cocycle is recurrent.

[1] We will also say that x is a transient (resp. recurrent) point for the cocycle.

[2] These assumptions will be satisfied in our applications.

Let (X,τ,ν) be a dynamical system, and let $\varphi : X \to G$ be a measurable function. Recall that φ is a *coboundary* if there exists a measurable function $\psi : X \to G$ such that $\varphi = \psi - \tau\psi$. Let $(\tilde{X},\tilde{\tau},\tilde{\nu})$ be the dynamical system defined by equation (2). Let $H \subset G$ be a closed subgroup; set $\tilde{X}_H = X \times G/H, \tilde{\nu}_H = \nu \times \mathrm{Leb}_{G/H}$. Define $\tilde{\tau}_H : \tilde{X}_H \to \tilde{X}_H$ by $\tilde{\tau}_H(x, g+H) = (\tau x, (\varphi(x)+g)+H)$. Then $(\tilde{X}_H,\tilde{\tau}_H,\tilde{\nu}_H)$ is the skew product over (X,τ,ν) with the fibre G/H and the displacement function $\varphi_H(x) = \varphi(x) + H$.

Lemma 2. *If there is a closed, proper subgroup $H \subset G$ such that the dynamical system $(\tilde{X}_H,\tilde{\tau}_H,\tilde{\nu}_H)$ is ergodic, then φ is not a coboundary.*

Proof. Assume the opposite, and let $\varphi = \psi - \tau\psi$ where $\psi : X \to G$ is a measurable function. Set $\Psi(x, g) = (x, \psi(x) + g)$. Then $\Psi : X \times G \to X \times G$ is an automorphism of the measure space $(X \times G, \tilde{\nu})$; it conjugates $\tilde{\tau}$ and the product transformation $\tau \times \mathrm{Id}$. Dividing by H, we obtain the automorphism $\Psi_H : X \times G/H \to X \times G/H$ conjugating $\tilde{\tau}_H$ and $\tau \times \mathrm{Id}_{G/H}$. This contradicts the ergodicity of $\tilde{\tau}_H$. ∎

We introduce a terminology to express the property that a walk in G visits any compact set with zero asymptotic frequency.

Definition 3. *Let (X,τ,ν) be a dynamical system with finite invariant measure, let $G \subset \mathbb{R}^d$ or $\mathbb{Z}^d$ be a closed subgroup, and let $\varphi : X \to G$ be a measurable function. The associated cocycle (φ_n) is* zero-recurrent *if it is recurrent, and for a.e. point $x \in X$ and any compact set $K \subset G$ we have*

$$\lim_{n\to\infty} \left\{ \frac{1}{n} \sum_{k=0}^{n-1} 1_K(\varphi(k,x)) \right\} = 0. \tag{5}$$

Proposition 1. *Let (X,τ,ν) be a dynamical system with finite invariant measure, let $\varphi : X \to \mathbb{R}$ be an integrable function, let (φ_n) be the associated cocycle, and let $(\tilde{X},\tilde{\tau},\tilde{\nu})$ be the corresponding skew product. Denote by $\mathcal{J}$ the σ-algebra of measurable τ-invariant subsets. Let $\mathbb{E}(\varphi|\mathcal{J})$ be the conditional expectation of φ with respect to $\mathcal{J}$. Let $R \subset X$ be the set of recurrent points for the cocycle (φ_n).*

Then the following properties hold.

1. *The set R and the set $\{x : \mathbb{E}(\varphi|\mathcal{J})(x) = 0\}$ coincide up to a set of ν-measure zero.*
2. *If the dynamical system (X,τ,ν) is ergodic and $\int_X \varphi \, d\nu = 0$, then the cocycle (φ_n) is recurrent.*
3. *If, moreover, φ is not a coboundary, then the cocycle (φ_n) is zero-recurrent.*

Proof. Let $A, B \subset X$ be measurable sets. By $A = B$ we will mean that A and B are equal in the measure-theoretic sense.[3] We will also say, simply, that A and B coincide.

The first two claims are classical. For ν-a.e. x the conditional expectation $\mathbb{E}(\varphi|\mathcal{J})(x)$ is defined, and, by the Birkhoff ergodic theorem, $\frac{1}{n}\varphi(n,x) \to \mathbb{E}(\varphi|\mathcal{J})(x)$. Hence, the τ-invariant sets $\{x : \frac{1}{n}\varphi(n,x) \to 0\}$ and $\{x : \mathbb{E}(\varphi|\mathcal{J})(x) = 0\}$ coincide. We denote this τ-invariant set by X_0. By Lemma 1, the cocycle (φ_n) is recurrent on X_0. By the Birkhoff ergodic theorem, almost every $x \in X \setminus X_0$ is transient for (φ_n); moreover, on $X \setminus X_0$, the cocycle has linear dissipation. Thus, $R = X_0 = \{x : \mathbb{E}(\varphi|\mathcal{J})(x) = 0\}$, proving claim 1. Claim 2 directly follows from claim 1. We will now prove claim 3.

Set $\tilde{X} = X \times \mathbb{R}$ and let $(\tilde{X}, \tilde{\tau}, \tilde{\nu})$ be the skew product equation (2). Let $K \subset \mathbb{R}$ be any compact. Set

$$u_K(x,g) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_K(\varphi(k,x) + g).$$

The ergodic theorem applied to $(\tilde{X}, \tilde{\tau}, \tilde{\nu})$ ensures the existence of the limit for a.e. (x,g) and that u_K is an integrable, nonnegative, $\tilde{\tau}$-invariant function on $\tilde{X}$. Suppose that $u_K \neq 0$ on a set of positive measure. Then $u_K\tilde{\nu}$ is a finite $\tilde{\tau}$-invariant measure on $\tilde{X}$, absolutely continuous with respect to $\tilde{\nu}$. Since (X, τ, ν) is ergodic, this implies that φ is a coboundary [5], contrary to the assumption. Thus, for any compact $K \subset \mathbb{R}$ we have $\lim \frac{1}{n}\sum_{k=0}^{n-1} 1_K(\varphi(k,x) + g) = 0$ for $\tilde{\nu}$-a.e. $(x,g) \in \tilde{X}$. Since $\mathbb{R}$ is a countable union of compacta, there exists $\tilde{Y} \subset \tilde{X}$, $\tilde{\nu}(\tilde{X} \setminus \tilde{Y}) = 0$, such that for $(x,g) \in \tilde{Y}$ and any compact $K \subset \mathbb{R}$ we have $u_K(x,g) = 0$.

By Fubini's theorem, there exists $g_0 \in \mathbb{R}$ such that, for ν a.e. $x \in X$ we have $\lim \frac{1}{n}\sum_{k=0}^{n-1} 1_K(\varphi(k,x) + g_0) = 0$ for any compact $K \subset \mathbb{R}$. But $g \mapsto g + g_0$ is a self-homeomorphism of $\mathbb{R}$. ∎

Remark 1. The same argument proves claim 3 for $\mathbb{R}^d$-valued cocycles. See [5] for a generalization to cocycles with values in locally compact groups.

Proposition 1 and Lemma 2 imply the following.

Corollary 1. *Let (X, τ, ν) be a dynamical system with finite measure. Let $\varphi : X \to \mathbb{R}$ be a measurable function such that $\int_X \varphi \, d\nu = 0$, and let*

[3] I. e., their symmetric difference is of ν-measure 0.

(φ_n) be the associated cocycle. Let $(\tilde{X}, \tilde{\tau}, \tilde{\nu})$ be the skew product corresponding to φ. For $p \in \mathbb{R}$ let $(\tilde{X}_p, \tilde{\tau}_p, \tilde{\nu}_p)$ be the reduction of $(\tilde{X}, \tilde{\tau}, \tilde{\nu})$ with respect to the subgroup $H = p\mathbb{Z}$, as in Lemma 2.

Suppose that for some $p \neq 0$ the dynamical system $(\tilde{X}_p, \tilde{\tau}_p, \tilde{\nu}_p)$ is ergodic. Then the cocycle (φ_n) is zero-recurrent.

We will use the following proposition.

Proposition 2. *Let (X, τ, ν) be an ergodic dynamical system with a finite measure, let $\varphi : X \to \mathbb{R}^d$ be a measurable function, and let $\varphi(n, x)$ be the corresponding cocycle. Then the following dichotomy holds: i) The cocycle (φ_n) is recurrent; ii) Let $k_n \in \mathbb{N}$ be any strictly increasing sequence. Then there exists $c > 0$[4] such that for a.e. $x \in X$ we have*

$$\limsup_n (n^{-1/d} |\varphi(k_n, x)|) = c. \tag{6}$$

Proof. The quantity $\limsup_n (n^{-\frac{1}{d}} |\varphi(k_n, x)|)$ is an invariant, measurable function. By ergodicity, it is equal to a constant $c \geq 0$. The claim is now immediate from Lemma 1. ∎

1.2. **Noncompact, periodic polygonal surfaces.**
We will now establish the geometric framework for our study. Let G be an infinitely countable group acting freely and cocompactly by isometries on a noncompact riemannian manifold $\tilde{P}$.[5] Then $P = \tilde{P}/G$ is compact; the projection $p : \tilde{P} \to P$ is a riemannian covering. Let $U\tilde{P}, UP$ be the unit tangent bundles for $\tilde{P}, P$; let $\tilde{T}^t, T^t$ be the respective geodesic flows; let $\tilde{\mu}, \mu$ be the Liouville measures for $U\tilde{P}, UP$ respectively. The action of G on $\tilde{P}$ uniquely extends to a free, cocompact action on $U\tilde{P}$. We have $UP = U\tilde{P}/G$; let $q : U\tilde{P} \to UP$ be the projection. Then $q : (U\tilde{P}, \tilde{T}^t, \tilde{\mu}) \to (UP, T^t, \mu)$ is a covering of flows.

Let $X \subset UP$ be a compact submanifold which is a cross-section for (UP, T^t, μ). Then the manifold $\tilde{X} = p^{-1}(X) \subset U\tilde{P}$ is a cross-section for the flow $(U\tilde{P}, \tilde{T}^t, \tilde{\mu})$. Let $\nu, \tilde{\nu}$ be the induced measures on $X, \tilde{X}$ respectively; let $\tau : X \to X$, $\tilde{\tau} : \tilde{X} \to \tilde{X}$ be the respective Poincaré maps. Then $\tilde{X} = X \times G$ measure theoretically, and $\tilde{\nu} = \nu \times \text{Leb}$. There is a unique mapping $\varphi : X \to G$ such that $(\tilde{X}, \tilde{\tau}, \tilde{\nu})$ is the skew product over (X, τ, ν) with the displacement function φ. See equation (2).

Let the manifold $\tilde{P}$ be a *noncompact polygonal surface.* See [10, 13] for the background. We will say that $\tilde{P}$ is a *G-periodic* polygonal surface. When the group G is implicit, we will say that $\tilde{P}$ is a periodic polygonal surface. When $G = \mathbb{Z}$ or $G = \mathbb{Z}^2$, we will say that $\tilde{P}$ is

[4] In general, it depends on the sequence.

[5] In general, with boundary and corners.

$\mathbb{Z}$-periodic or $\mathbb{Z}^2$-periodic respectively. The projection $p : \tilde{P} \to P$ is a *covering of polygonal surfaces* [13].

If $\tilde{P}$ (or, equivalently, P) is a *rational polygonal surface*, we associate with it a finite subgroup $\Gamma = \Gamma(P) \subset O(2)$. For $N \geq 1$ let $R_N \subset O(2)$ be the dihedral group of order $2N$, i. e., the group generated by two orthogonal reflections, with the angle π/N between their axes. If $\partial P \neq \emptyset$, then $\Gamma(P) = R_N$, where N is determined by P. Note that R_1 consists of a reflection and the identity. The associated *translation surface* $S = S(P)$ [14] is a compact riemann surface endowed with a riemannian metric, flat everywhere except for a finite number of *cone points*. The group $\Gamma(P)$ acts on $S(P)$ by isometries, and we have $P = S(P)/\Gamma(P)$ [10]. Thus, a polygonal surface is a translation surface iff $\Gamma(P) = \mathrm{Id}$. Let $\Gamma = \Gamma(P)$ and let $S = S(P)$. Then there is a unique noncompact, G-periodic translation surface $\tilde{S}$ such that the following conditions hold. The groups Γ and G act on $\tilde{S}$ by isometries; the two actions commute. We have $\tilde{S}/G = S$, $\tilde{S}/\Gamma = \tilde{P}$, $S/\Gamma = P$, $\tilde{P}/G = P$; the projections $\tilde{S} \to S$, $\tilde{P} \to P$, $\tilde{S} \to \tilde{P}$, $S \to P$ are compatible.

The group Γ acts on the unit circle $U \subset \mathbb{R}^2$; let U/Γ be the quotient. The flows $(U\tilde{P}, \tilde{T}^t, \tilde{\mu})$ and (UP, T^t, μ) decompose as one-parameter families of *directional geodesic flows* $(U\tilde{P}_\theta, \tilde{T}^t_\theta, \tilde{\mu}_\theta)$ and $(UP_\theta, T^t_\theta, \mu_\theta)$ where $\theta \in U/\Gamma$. The projection q is compatible with the decompositions, inducing the directional projections $q_\theta : (U\tilde{P}_\theta, \tilde{T}^t_\theta, \tilde{\mu}_\theta) \to (UP_\theta, T^t_\theta, \mu_\theta)$. The flow $(UP_\theta, T^t_\theta, \mu_\theta)$ is naturally isomorphic to the linear flow in direction θ on the translation surface S; the measure μ_θ corresponds to the Lebesgue measure on S. Analogously, $(U\tilde{P}_\theta, \tilde{T}^t_\theta, \tilde{\mu}_\theta)$ is the *linear flow in direction θ on the periodic translation surface $\tilde{S}$* [13].

Example 1. Let $B \subset \mathbb{R}^2$ be the horizontal strip bounded by the lines $\{y = 0\}$ and $\{y = 1\}$. For $0 \leq a, b < 1, a + b > 0$ let $R_0 = R_0(a, b)$ be the $(a \times b)$-rectangle centered at $(1/2, 1/2)$. Set $\tilde{P}(a, b) = B \setminus \cup_{k \in \mathbb{Z}}(R_0 + (k, 0))$. The $\mathbb{Z}$-periodic polygonal surface $\tilde{P}(a, b)$ is a strip with a periodic sequence of rectangular obstacles. See figure 2. Let Q be the unit square $0 \leq x, y \leq 1$; let C be the cylinder obtained by identifying the vertical sides of Q. Then $P(a, b) = C \setminus R_0(a, b)$ is the unit cylinder with a rectangular obstacle. In the limit cases $a = 0$ or $b = 0$ the obstacles degenerate into *barriers*. We have $\Gamma = R_2$ if $b \neq 0$ and $\Gamma = R_1$ if $b = 0$.

Set $P = P(a, b)$ and $S = S(a, b)$. Let $a, b \neq 0$. The translation surface S is constructed from 4 copies of $Q \setminus R_0(a, b)$ via identifications of their sides shown in figure 3. It has 4 cone points with cone angles 6π.

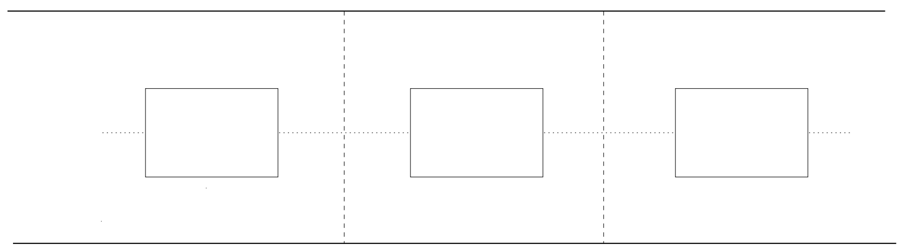

FIGURE 2. *An infinite band with a periodic configuration of rectangular obstacles.*

This yields $g(S) = 5$ [14]. The genus of S can also be computed directly from the angles in P [10]. The same analysis applies when $a = 0$ or $b = 0$. If $a = 0$, the surface S is made from 4 rectangles with vertical barriers. It has 4 cone points with cone angles 4π, yielding $g(S) = 3$. If $b = 0$, then S is made from 2 rectangles with horizontal barriers. There are 2 cone points with cone angles 4π, yielding $g(S) = 2$.

Set $\tilde{P} = \tilde{P}(a, b)$ and $\tilde{S} = \tilde{S}(a, b)$. The noncompact translation surface $\tilde{S}$ is obtained by analogous identifications of pairs of sides in the disjoint union of 4 copies of $\tilde{P}$. Since $\tilde{P}$ is $\mathbb{Z}$-periodic, and since these identifications are compatible with the action of $\mathbb{Z}$, the translation surface $\tilde{S}$ is $\mathbb{Z}$-periodic. It has infinite genus.

1.3. **The billiard flow and the billiard map.**

Let P be a compact polygonal surface, and let ∂P be its boundary. Orbits of the geodesic flow (UP, T^t, μ), viewed as curves in P, are the geodesics. We will use the term *billiard curves* for those geodesics that intersect ∂P at regular points.

Virtually every polygonal surface has singular points. Points $z \in \text{interior}(P)$ (resp. $z \in \partial(P)$) are singular if they are cone points (resp. corner points). A geodesic in P may start or end at a singular point, but it cannot pass through a singular point. A phase point $v \in UP$ is singular if the geodesic it defines arrives at a singular point, and hence is defined on a proper subinterval of $(-\infty, \infty)$. Therefore the geodesic flow $T^t, -\infty < t < \infty$, is defined only on the set of regular (i. e., non-singular) points in UP. We will use the terms *regular and singular sets* for the sets of regular and singular points, respectively.

The singular set has codimension one; thus, the Liouville measure of the singular set is zero, and the regular set has full measure. As is usual in the billiard literature [10, 15], we will use the notation (UP, T^t, μ) for the geodesic flow on the regular set. Analogous notational conventions are used for the billiard map, which we will now define. The reader

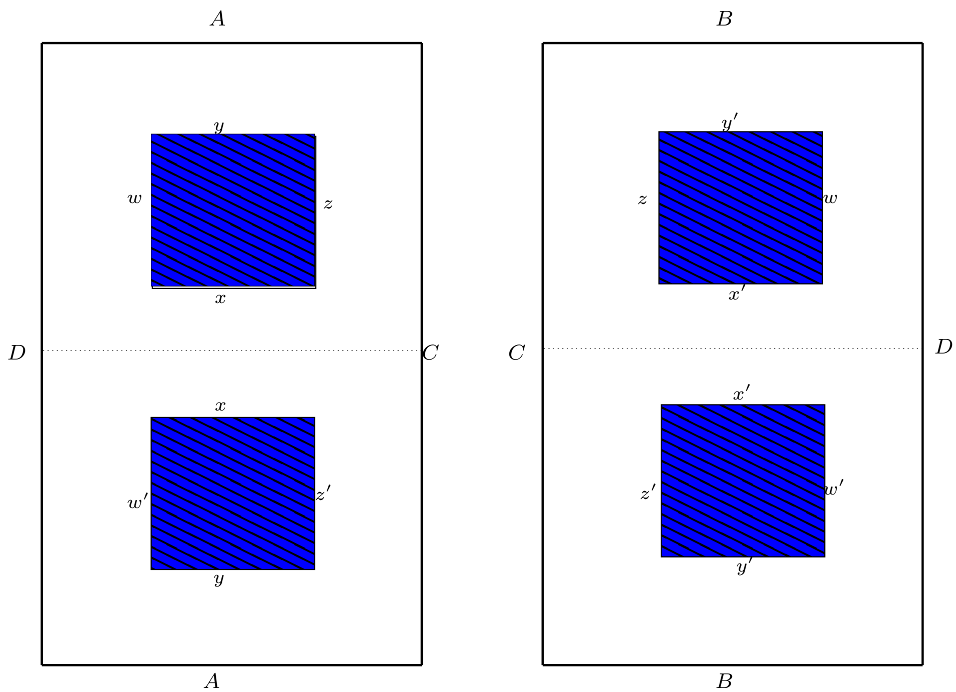

FIGURE 3. *The translation surface made from 4 copies of $Q \setminus R_0(a, b)$ by identifying the sides bearing the same labels.*

should mentally substitute "regular phase points" whenever we speak of phase points in what follows. The billiard map will be defined on the regular set which has full measure in the canonical cross-section that we will now describe.

Let $A \subset P$. We denote by $U_A P \subset UP$ the set of vectors $v \in UP$ whose base points belong to A. Let $BUP \subset UP$ be the smallest T^t-invariant set containing $U_{\partial P} P$. If $\partial P \neq \emptyset$, we set $\mu_B = \mu|_{BUP}$. By definition, $U_{\partial P} P$ is a cross-section for the flow (BUP, T^t, μ_B). Let ν be the induced measure on $U_{\partial P} P$. We will use the following terminology: The flow (BUP, T^t, μ_B) is the *billiard flow* of P; the set $U_{\partial P} P \subset BUP$ is the *standard cross-section* for the billiard flow; the induced transformation $(U_{\partial P} P, \tau, \nu)$ is the *billiard map* for P.

Let now P be a rational polygonal surface with a boundary, and let $\Gamma(P) = R_N$. We identify U/Γ and $[0, \pi/N]$. For $\theta \in [0, \pi/N]$ set $U_{\partial P} P_\theta = UP_\theta \cap U_{\partial P} P$, $BUP_\theta = BUP \cap UP_\theta$. Suppose that $\mu_\theta(BUP_\theta) > 0$. Let $b\mu_\theta$ be the restriction of μ_θ to BUP_θ; then $(BUP_\theta, T^t_\theta, b\mu_\theta)$ is the *billiard flow in direction* θ. The set $U_{\partial P} P_\theta \subset BUP_\theta$ is the *standard directional cross-section*. Let ν_θ be the induced measure on $U_{\partial P} P_\theta$.

The induced dynamical system $(U_{\partial P}P_\theta, \tau_\theta, \nu_\theta)$ is the *directional billiard map* [13]. When $P \subset \mathbb{R}^2$ is a compact polygon, this is the standard terminology [12].

Let S be a compact translation surface. Let $O \subset S$ be a polygon, not necessarily connected, such that $S \setminus O$ is connected. The polygonal surface $P = S \setminus \text{interior}(O)$ is a *translation surface with polygonal obstacles*. Some of the components of O may be linear segments, hence we will also speak of *translation surface with polygonal obstacles and/or barriers*.

Lemma 3. *1. Let P be a compact translation surface with polygonal obstacles and/or barriers. Then $\mu(UP \setminus BUP) = 0$. 2. Let P be a compact, rational polygonal surface with a boundary; let R_N be the corresponding reflection group. Let $\mathcal{E}_{erg}(P) \subset [0, \pi/N]$ be the set of uniquely ergodic directions. i) If ∂P contains intervals with distinct directions, then for every $\theta \in \mathcal{E}_{erg}(P)$ we have $BUP_\theta = UP_\theta$. ii) Suppose that ∂P consists of intervals with the same direction, say θ_0. Then for $\theta \in \mathcal{E}_{erg}(P) \setminus \{\theta_0\}$ we have $BUP_\theta = UP_\theta$.*

Proof. 1. Let $P = S \setminus O$. Let γ be an infinite geodesic in P that does not intersect O. Then γ is an infinite geodesic in S, and $\gamma \cap O = \emptyset$. If $\text{interior}(O) \neq \emptyset$, then γ is not dense, and hence its direction is not minimal. Suppose $\text{interior}(O) = \emptyset$, i. e., O consists of barriers. Assume that O contains segments of distinct directions. A geometric argument which we leave to the reader implies that γ is not dense in S, and hence its direction is not minimal. Let O consist of segments with direction θ_0. Then the direction of γ is either nonminimal or it is θ_0. But the set of nonminimal directions is countable, implying the claim.

2. There is a compact translation surface with obstacles and/or barriers, say $S \setminus O$, and a finite covering $p : (S \setminus O) \to P$. Let $\beta \subset P$ be an infinite geodesic in direction θ, and let $\alpha \subset S$ be its pull back by p. The preceding argument shows that in the case i) (resp. case ii)) the direction of α belongs to $U \setminus \mathcal{E}_{\text{erg}}$ (resp. $U \setminus (\mathcal{E}_{\text{erg}} \cup \{\theta_0\})$). ■

Let $\tilde{P}$ be a noncompact, G-periodic polygonal surface, and let $P = \tilde{P}/G$ be the compact quotient. Suppose that $\partial P \neq \emptyset$. By Lemma 3, $U_{\partial P}P$ and $U_{\partial \tilde{P}}\tilde{P}$ are the cross-sections for the billiard flows of P and $\tilde{P}$ respectively. Let $(U_{\partial P}P, \tau, \nu)$ and $(U_{\partial \tilde{P}}\tilde{P}, \tilde{\tau}, \tilde{\nu})$ be the respective billiard maps. Then $(U_{\partial \tilde{P}}\tilde{P}, \tilde{\tau}, \tilde{\nu})$ is a skew product over $(U_{\partial P}P, \tau, \nu)$ with the fibre G and a displacement function $\varphi : U_{\partial P}P \to G$. See equation (2).

Lemma 4. *The displacement function is centered:*

$$\int_{U_{\partial P}P} \varphi d\nu = 0. \tag{7}$$

Proof. For $v \in U_{\partial P}P$ let $\gamma_v = \{\gamma_v(t) : 0 \leq t \leq t(v)\}$ be the segment of the geodesic ray $\{\gamma_v(t)\}$ determined by v ending when $\{\gamma_v\}$ first returns to ∂P. Note that $t(v)$ is the *first return time function* for the billiard map. The tangent vector $\gamma_v'(t)$ is defined for $0 < t < t(v)$ and the left limit $\lim_{t\to t(v)-} \gamma_v'(t)$ exists. Set $\sigma(v) = -\lim_{t\to t(v)-} \gamma_v'(t)$. The transformation $\sigma : U_{\partial P}P \to U_{\partial P}P$ is the *canonical involution* for the billiard map $(U_{\partial P}P, \tau, \nu)$ [12].

The canonical involution $\tilde{\sigma}$ for the billiard map $(U_{\partial\tilde{P}}\tilde{P}, \tilde{\tau}, \tilde{\nu})$ is defined the same way. Let $\tilde{v} = (v, g) \in U_{\partial\tilde{P}}\tilde{P}$. Then

$$\tilde{\sigma}(\tilde{v}) = \tilde{\sigma}(v, g) = (\sigma(v), g + \varphi(v)).$$

The identity

$$(v, g) = \tilde{\sigma}^2(v, g) = \tilde{\sigma}(\sigma(v), g + \varphi(v)) = (\sigma^2(v), g + \varphi(v) + \varphi(\sigma(v)))$$

yields

$$\varphi(\sigma(v)) = -\varphi(v). \tag{8}$$

Since σ preserves the Liouville measure, the claim follows. ■

Let now $\tilde{P}$ be a rational polygonal surface. Let $\Gamma(P) = R_N$; we identify $U/\Gamma(P)$ and $[0, \pi/N]$. For $\theta \in [0, \pi/N]$ let $(U_{\partial P}P_\theta, \tau_\theta, \mu_\theta)$ and $(U_{\partial\tilde{P}}\tilde{P}_\theta, \tilde{\tau}_\theta, \tilde{\mu}_\theta)$ be the directional billiard maps for P and $\tilde{P}$ respectively. Then $(U_{\partial\tilde{P}}\tilde{P}_\theta, \tilde{\tau}_\theta, \tilde{\mu}_\theta)$ is the skew product over $(U_{\partial P}P_\theta, \tau_\theta, \mu_\theta)$ with the displacement function $\varphi_\theta = \varphi|_{U_{\partial P}P_\theta}$. By equation (2)

$$\tilde{\tau}_\theta(v, g) = (\tau_\theta(v), g + \varphi_\theta(v)). \tag{9}$$

Lemma 5. *Let N be even. Then for every $\theta \in [0, \pi/N]$ the* directional displacement function φ_θ *is centered:*

$$\int_{U_{\partial P}P_\theta} \varphi_\theta d\nu_\theta = 0. \tag{10}$$

If N is odd, then the function $\varphi_{\pi/(2N)}$ is centered.

Proof. The canonical involution $\sigma : U_{\partial P}P \to U_{\partial P}P$ induces *directional involutions* $\sigma_\theta : U_{\partial P}P_\theta \to U_{\partial P}P_{\eta(\theta)}$. The central symmetry of U and the identification $U/R_N = [0, \pi/N]$ induce the transformation $\theta \mapsto \eta(\theta)$ of $[0, \pi/N]$. The proof of Lemma 4 yields

$$\int_{U_{\partial P}P_{\eta(\theta)}} \varphi_{\eta(\theta)} d\nu_{\eta(\theta)} = -\int_{U_{\partial P}P_\theta} \varphi_\theta d\nu_\theta. \tag{11}$$

If N is even, then R_N contains the central symmetry, hence $\eta(\theta) = \theta$. If N is odd, then $\eta(\theta) = \pi/N - \theta$. Both claims now follow from equation (11). ∎

2. $\mathbb{Z}$-periodic polygonal surfaces

We will use the setting and the notation of section 1, with $G = \mathbb{Z}$. Let $\tilde{P}$ be a $\mathbb{Z}$-periodic polygonal surface, and let $P = \tilde{P}/\mathbb{Z}$. If P is a rational polygonal surface, we will denote by $\tilde{S}$ and S the translation surfaces of $\tilde{P}$ and P respectively. Then $\tilde{S}$ is $\mathbb{Z}$-periodic, and $S = \tilde{S}/\mathbb{Z}$.

2.1. **Main result.**

A compact translation surface S is *arithmetic* [14] if it admits a translation covering $\pi : S \to \mathbb{T}^2$ onto a flat torus whose branch locus is a single point. Via an affine renormalization, we can assume that S covers the standard torus $\mathbb{T}^2_0 = \mathbb{R}^2/\mathbb{Z}^2$ and that the branch locus is $\{0\} + \mathbb{Z}^2$. These translation surfaces are also known as *square-tiled* and as *origamis*.

The surface $\tilde{P}$ is arithmetic iff $P = \tilde{P}/\mathbb{Z}$ is arithmetic [13]. Let P be a compact, arithmetic polygonal surface; let $\pi : S \to \mathbb{T}^2_0$ be as above. Let $\Gamma = \Gamma(P)$. A direction $\theta \in U$ is rational if $\tan\theta \in \mathbb{Q}$. Using the covering $\pi : S \to \mathbb{T}^2$ and the natural action of $\mathrm{GL}(2, \mathbb{R})$ on translation surfaces [14], we extend the notion of *rational directions* to all arithmetic translation surfaces, and hence to arithmetic polygonal surfaces.[6]

Let $(U/\Gamma)_{\mathrm{rat}} \subset U/\Gamma$ be the set of P-rational directions. Then $\theta \in (U/\Gamma)_{\mathrm{rat}}$ iff every geodesic in P in direction θ is periodic or a *saddle connection* [10]. The set $(U/\Gamma)_{\mathrm{rat}}$ is countable. We set $(U/\Gamma)_{\mathrm{irr}} = U/\Gamma \setminus (U/\Gamma)_{\mathrm{rat}}$; we say that $\theta \in (U/\Gamma)_{\mathrm{irr}}$ are the *irrational directions*.

Theorem 1. *Let $\tilde{P}$ be a $\mathbb{Z}$-periodic polygonal surface with a boundary, and let $P = \tilde{P}/\mathbb{Z}$.*
1. If the flow (UP, T^t, μ) is ergodic, then the geodesic flow for $\tilde{P}$ is recurrent.
2. Let P be a rational polygonal surface, and let $\Gamma = \Gamma(P)$. Suppose that $|\Gamma|$ is divisible by 4. Then for a full measure set of directions $\theta \in U/\Gamma$ the directional geodesic flow $(U\tilde{P}_\theta, \tilde{T}^t_\theta, \tilde{\lambda}_\theta)$ is zero-recurrent. (See definition 3.)
3. Let P be an arithmetic polygonal surface. i) For an irrational direction θ the flow $(U\tilde{P}_\theta, \tilde{T}^t_\theta, \tilde{\lambda}_\theta)$ is zero-recurrent. ii) Let θ be a rational direction. Then the set of orbits of $\tilde{T}^t_\theta$ is a disjoint union of periodic

[6] We will say *P-rational* to emphasize that the set of rational directions depends on the surface in question.

bands and bands of orbits that are dissipative with a positive rate. The boundaries of these bands are concatenations of saddle connections.

Proof. 1. The claim follows from Lemma 4 and claim 2 in Proposition 1.
2. Let $\theta \in \mathcal{E}_{\mathrm{erg}}(P)$, the set of uniquely ergodic directions. By Lemma 3, Lemma 5, and Proposition 1, the flow $(U\tilde{P}_\theta, \tilde{T}_\theta^t, \tilde{\lambda}_\theta)$ is conservative. Since the set $\mathcal{E}_{\mathrm{erg}}(P) \subset S^1$ has full lebesgue measure [19], we obtain that $(U\tilde{P}_\theta, \tilde{T}_\theta^t, \tilde{\lambda}_\theta)$ is conservative for a.e. θ.

For $k \in \mathbb{N}$ set $P_k = \tilde{P}/k\mathbb{Z}$. Let $l, k \in \mathbb{N}$ and let ℓ divide k. Then there is a covering $p_{k,\ell} : P_k \to P_\ell$, implying $\mathcal{E}_{\mathrm{erg}}(P_k) \subseteq \mathcal{E}_{\mathrm{erg}}(P_\ell)$. In particular, $\mathcal{E}_{\mathrm{erg}}(P_k) \subseteq \mathcal{E}_{\mathrm{erg}}(P)$ for any $k > 1$. By Lemma 2 and Proposition 1, if $\theta \in \mathcal{E}_{\mathrm{erg}}(P_k)$, and $k > 1$, then the flow $(U\tilde{P}_\theta, \tilde{T}_\theta^t, \tilde{\lambda}_\theta)$ is zero-recurrent. Thus, $(U\tilde{P}_\theta, \tilde{T}_\theta^t, \tilde{\lambda}_\theta)$ is zero-recurrent for $\theta \in \cup_{k>1}\mathcal{E}_{\mathrm{erg}}(P_k)$, a full measure subset of $\mathcal{E}_{\mathrm{erg}}(P)$.
3. In this case all surfaces P_k are arithmetic and $\mathcal{E}_{\mathrm{erg}}(P_k) = \mathcal{E}_{\mathrm{erg}}(P) = (U/\Gamma)_{\mathrm{irr}}$ [10]. We can assume without loss of generality that $\mathcal{E}_{\mathrm{erg}}(P) = [0, \pi/N] \setminus \mathbb{Q}$. The preceding argument yields claim i).

Let now $\theta \in [0, \pi/N] \cap \mathbb{Q}$. By [10], the flow (UP, T_θ^t, μ) decomposes into periodic bands whose boundaries are made from saddle connections. Depending on whether ergodic sums of the displacement function along a periodic orbit vanish or not, the preimage of a periodic band in $U\tilde{P}$ is a union of periodic and transient bands. Claim ii) follows. ∎

Corollary 2. *Let $\tilde{P}$ be a $\mathbb{Z}$-periodic, rational polygonal surface with a boundary. If $|\Gamma(P)|$ is divisible by 4, then the flow $(U\tilde{P}, \tilde{T}^t, \tilde{\mu})$ is zero-recurrent.*

Proof. Follows from the decomposition of $(U\tilde{P}, \tilde{T}^t, \tilde{\mu})$ into the directional flows $(U\tilde{P}_\theta, \tilde{T}_\theta^t, \tilde{\lambda}_\theta)$, a Fubini-type argument, and claim 2 in Theorem 1. ∎

2.2. **Examples and applications.**

We will now illustrate the preceding material with a few examples.

Example 2. Let $0 < h < 1$ and $0 \le a, b < 1$ be such that $0 < h \pm \frac{b}{2} < 1$ and $a + b > 0$. Let $R(a, b; h) \subset \mathbb{R}^2$ be the closed $a \times b$ rectangle centered at $(\frac{1}{2}, h)$, whose sides are parallel to the coordinate axes. Then $R(a, b; h)$ belongs to the interior of the unit square $Q = \{(x, y) : 0 \le x, y \le 1\}$. Let $P(a, b; h)$ be the polygonal surface obtained by deleting from Q the interior of $R(a, b; h)$, and identifying the sides $\{x = 0\}, \{x = 1\}$. If $0 < a, b$, then $P(a, b; h)$ is the flat unit cylinder with a rectangular obstacle. The obstacle is the $a \times b$ rectangle centered in the cylinder at the height h. See figure 4. If $b = 0$ (resp. $a = 0$) then

the rectangular obstacle degenerates into a horizontal (resp. vertical) barrier.

For $k \in \mathbb{Z}$ let $R_k(a,b;h) = R(a,b;h) + (k,0)$; let $B = \{(x,y) : -\infty < x < \infty, 0 \leq y \leq 1\}$. Set $\tilde{P}(a,b;h) = B \setminus \cup_{k\in\mathbb{Z}} R_k(a,b;h)$. Then $\tilde{P}(a,b;h)$ is a $\mathbb{Z}$-periodic polygonal surface, and $P(a,b;h) = \tilde{P}(a,b;h)/\mathbb{Z}$. When $h = \frac{1}{2}$, we recover Example 1.

Let $\Gamma = \Gamma(P(a,b;h))$. If $b \neq 0$, then $|\Gamma| = 4$; when $b = 0$, then $|\Gamma| = 2$. Thus, for $b \neq 0$ the surface $\tilde{P}(a,b;h)$ satisfies the assumptions of claim 2 in Theorem 1. The surface $P(a,b;h)$ is arithmetic iff $a, b \in \mathbb{Q}$ [14]. Theorem 1 and Corollary 2 imply the following statement.

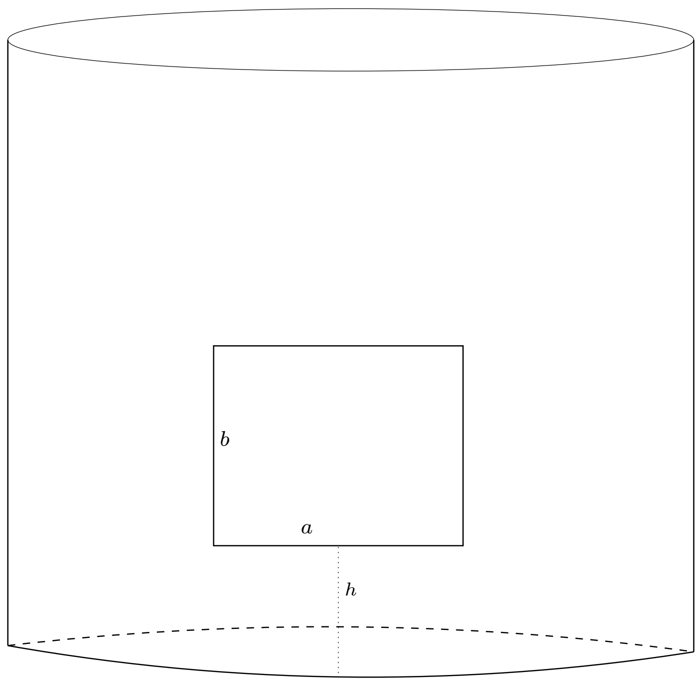

FIGURE 4. *Flat cylinder with a rectangular obstacle.*

Corollary 3. *Let* $(U\tilde{P}(a,b;h), \tilde{T}^t, \tilde{\mu})$ *be the geodesic flow for* $\tilde{P}(a,b;h)$*; let* $(U\tilde{P}(a,b;h)_\theta, \tilde{T}^t_\theta, \tilde{\mu}_\theta)$ *be the directional flows. We will refer to them as* $\tilde{T}^t$ *and* $\tilde{T}^t_\theta$*. Let* $b \neq 0$*. Then the following claims hold.*
1. The flow $\tilde{T}^t$ *is zero-recurrent.*
2. For a.e. $\theta \in [0, \pi/2]$ *the flow* $\tilde{T}^t_\theta$ *is zero-recurrent.*
3. Let $a, b, h \in \mathbb{Q}$*. Then, for every* $\theta \in [0, \pi/2]$ *such that* $\tan\theta \notin \mathbb{Q}$*, the flow* $\tilde{T}^t_\theta$ *is zero-recurrent.*

Remark 2. If $b = 0$, then $\tilde{P}(a, 0; h)$ is the horizontal band with a periodic configuration of horizontal barriers of length a. Thus $N = 1$ and $U/\Gamma = [0, \pi]$. See Example 15 in [13]. For $\theta \neq \pi/2$ the flows $\tilde{T}_\theta^t$ are transient: Every orbit of $\tilde{T}_\theta^t$ drifts horizontally with the rate $\sin\theta$. The flow $\tilde{T}_{\pi/2}^t$ is periodic. This example fits into the framework of Lemma 5.

Let $Q \subset \mathbb{R}^2$ be a polygon satisfying for an integer $t \geq 1$ the following conditions. i) There is a nonzero vector $\vec{v} \in \mathbb{R}^2$, and for $1 \leq i \leq t$ there are sides s_i, s_i' of Q such that $s_i' = s_i + \vec{v}$. ii) We have $Q \cap (Q + \vec{v}) = \cup_{1 \leq i \leq t} s_i'$. We denote by $\tilde{P} = \tilde{P}(Q)$ the $\mathbb{Z}$-periodic polygon obtained by deleting from $\cup_{k\in\mathbb{Z}}(Q + k\vec{v})$ the sides of the form $s_i + k\vec{v} : 1 \leq i \leq t, k \in \mathbb{Z}$. We say that $\tilde{P}$ is the *stairway based on* Q or, simply, a *stairway*. The compact, polygonal surface $P = \tilde{P}/\mathbb{Z}$ is obtained by identifying the sides s_i and s_i' of Q for $1 \leq i \leq t$. Since $\cup_{1\leq i\leq t}(s_i \cup s_i') \subset \partial Q$ is a proper subset, $\partial P \neq \emptyset$. Let $\Gamma \subset O(2)$ be the group generated by reflections about the sides of Q other than $s_i, s_i' : 1 \leq i \leq t$. If $|\Gamma| < \infty$, then $\tilde{P}$ is a *rational stairway*. The following is immediate from claims 2 and 3 in Theorem 1.

Theorem 2. *Let $\tilde{P} \subset \mathbb{R}^2$ be a rational stairway, and let $\Gamma = R_N$. If N is even, then the following claims hold.*
1. For a full measure set of directions the flow $(U\tilde{P}_\theta, \tilde{T}_\theta^t, \tilde{\mu}_\theta)$ is zero-recurrent.
2. Suppose, in addition, that the surface $\tilde{P}$ is arithmetic. Then for every irrational direction the flow $(U\tilde{P}_\theta, \tilde{T}_\theta^t, \tilde{\mu}_\theta)$ is zero-recurrent.

Example 3. Let $a, b > 0$. Let $Q = Q(a, b)$ be the $2a \times b$ rectangle. We view ∂Q as a union of 6 sides: 2 vertical sides of length b and 4 horizontal sides of length a. Let s, s' be the lower left and the upper right horizontal sides respectively. Then $\tilde{P}(a, b)$ based on Q is the infinite stairway, with the stairs of length a and height b. Its quotient $P(a, b)$ is the rectangle Q with two sides of length a identified. See figure 5. The corresponding group is R_2. By [10, 14], $\tilde{P}(a, b)$ is arithmetic iff $a, b \in \mathbb{Q}$.

Corollary 4. *Let $\tilde{P}(a, b)$ be the stairway in Example 3. Then for a.e. $\theta \in [0, \pi/2]$ the flow $\tilde{T}_\theta^t$ is zero-recurrent. If $a, b \in \mathbb{Q}$, then $\tilde{T}_\theta^t$ is zero-recurrent if* $\tan\theta \notin \mathbb{Q}$.

The claims of Corollary 4 are immediate, by Theorem 2. See [17] for another proof of recurrence of $\tilde{T}_\theta^t$ and [18] for a study of ergodic invariant measures.

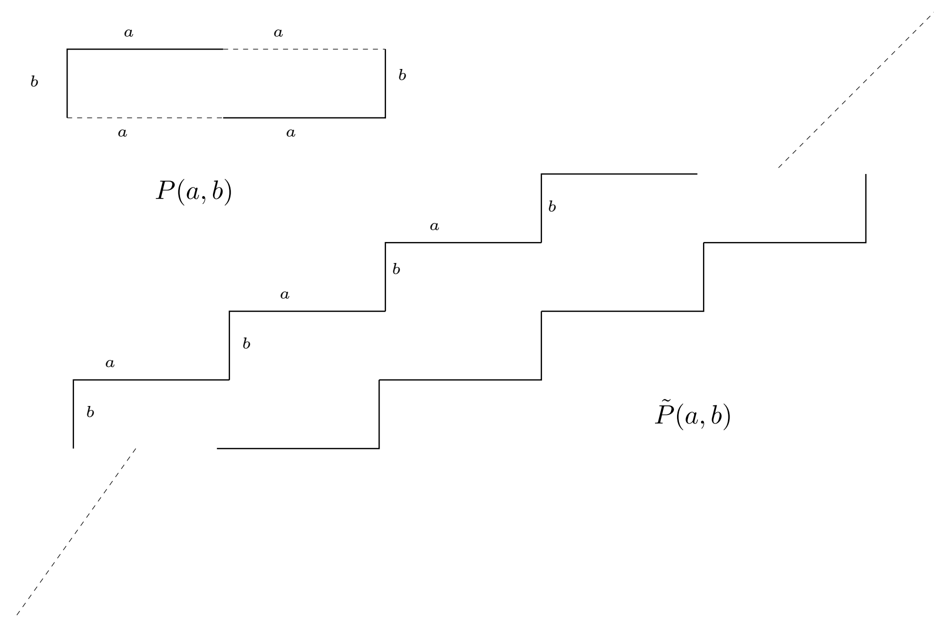

FIGURE 5. *A stairway polygonal surface and its quotient.*

2.3. **Generalizations and further applications.**
Let R_0 be the unit square; let $O \subset R_0$ be a polygon such that $R_0 \setminus O$ is connected. Then $\tilde{P}_O = \cup_{k\in\mathbb{Z}}\{(R_0 \setminus \text{interior}(O)) + (k,0)\}$ is a periodic band with obstacles and/or barriers. We will study the recurrence for the billiard in $\tilde{P}_O$.

Let $R(a,b;h;\alpha)$ be the rectangle in Example 2 rotated by α about its center point. We assume that a, b, h are such that $R(a,b;h;\alpha)$ belongs to the interior of the unit square for all α. Set $\tilde{P}(\alpha) = \tilde{P}_{R(a,b;h;\alpha)}$.

Recall that a subset in a topological space is *residual* if it contains a dense G_δ set.

Proposition 3. *The set of $\alpha \in S^1$ such that the flow $(U\tilde{P}(\alpha), \tilde{T}^t, \tilde{\mu})$ is recurrent is residual.*

Proof. Set $P(\alpha) = \tilde{P}(\alpha)/\mathbb{Z}$. By [19], the billiard flow for $P(\alpha)$ is ergodic for a dense G_δ set of angles α. The statement now follows from claim 1 in Theorem 1. ∎

The set of planar polygons has a natural topology [12]. In this topology, polygons with a fixed number of sides form closed subsets in euclidean spaces. Imposing upper bounds on the sizes of polygons, we

obtain (relatively) compact subsets in euclidean spaces. In what follows, whenever we invoke topological notions for spaces of polygons, we mean the natural topology.

Proposition 4. *Let $\mathcal{T}$ be the space of triangles in the interior of the unit square. For $O \in \mathcal{T}$ let $\tilde{P}_O$ be the corresponding periodic band with triangular obstacles. Then the set of triangles such that the flow $(U\tilde{P}_O, \tilde{T}^t, \tilde{\mu})$ is recurrent is residual.*

Proof. The space $\mathcal{T}$ is a relatively compact subset in $\mathbb{R}^6$. For $O \in \mathcal{T}$ the quotient $P_O = \tilde{P}_O/\mathbb{Z}$ is the standard cylinder with a triangular obstacle. By [19], $\mathcal{T}$ contains a dense G_δ set of triangles such that the geodesic flow on P_O is ergodic. Now we apply claim 1 in Theorem 1. ∎

Let $\mathcal{C}$ be a closed set of polygons inside the unit square. For $O \in \mathcal{C}$ let $\tilde{P}_O$ be the corresponding periodic band with obstacles. Propositions 3 and 4 are special cases of the following.

Theorem 3. *The set of $O \in \mathcal{C}$ such that the geodesic flow on $\tilde{P}_O$ is recurrent is residual.*

The proof of Theorem 3 is analogous to the proofs of Propositions 3, 4. It invokes the result in [19] that the ergodicity is topologically typical. It is not known if ergodicity is typical measure theoretically [12].

Y. Vorobets found a sufficient condition for the ergodicity of a (compact) polygon [23]. The condition invokes the speed of approximation of π-irrational angles of P by rationals. Referring the reader to [23] for a precise formulation, we will say that the angles *admit the Vorobets approximation.*

Theorem 4. *Let O be a polygon inside the unit square. Suppose that all irrational angles of O and those between O and the horizontal axis admit the Vorobets approximation. Suppose, moreover, that not all of these angles are π-rational. Let $\tilde{P}_O$ be the corresponding periodic band. Then the geodesic flow on $\tilde{P}_O$ is zero-recurrent.*

Proof. For $k \in \mathbb{N}$ set $P_k = \tilde{P}_O/k\mathbb{Z}$. Since all irrational angles of P_k admit the Vorobets approximation, the billiard flow on P_k is ergodic. If l divides k, we have the covering $p_{k,l} : P_k \to P_l$. It remains to invoke the proof of claim 2 in Theorem 1. ∎

We will now define a property of cocycles that has a simple geometric meaning. Let (X, τ, ν) be a dynamical system with a finite invariant measure. Let $\varphi : X \to \mathbb{R}$ be a measurable function, and let (φ_n) be the corresponding cocycle.

Definition 4. *The cocycle* (φ_n) *has (the property of) unbounded oscillations if for a. e.* $x \in X$ *we have*

$$\sup_n \varphi(n,x) = +\infty, \ \inf_n \varphi(n,x) = -\infty. \tag{12}$$

Let $(\tilde{X}, \tau_\varphi, \tilde{\tau})$ be the skew product over (X, τ, ν) with the displacement function φ. If the cocycle (φ_n) has unbounded oscillations, then it is recurrent. Suppose that $(\tilde{X}, \tau_\varphi, \tilde{\tau})$ is the Poincaré map of a skew product flow. Then a. e. orbit of the flow has unbounded oscillations in the obvious sense. See [3] for examples of physical systems corresponding to the billiard with unbounded oscillations in periodic polygons.

Proposition 5. *Let* (X, τ, ν) *be an ergodic dynamical system with finite invariant measure. Let* $\varphi : X \to \mathbb{R}$ *be a measurable function satisfying* $\int_X \varphi \, d\nu = 0$*; let* (φ_n) *be the corresponding cocycle. If* φ *is not a coboundary, then* (φ_n) *has unbounded oscillations.*

Proof. Suppose that the property $\sup_n \varphi(n,x) = +\infty$ for a. e. $x \in X$ is not satisfied. Then, by ergodicity, $\sup_n \varphi(n,x) < \infty$ for a. e. $x \in X$. Set

$$h(x) = \sup_{k \geq 1} \varphi(k,x), \ g(x) = \sup_{k \geq 2} \varphi(k,x) - h(x). \tag{13}$$

Since $\tau\varphi(k,x) = \varphi(k+1,x) - \varphi(x)$, we have

$$\varphi(x) = \sup_{k \geq 2} \varphi(k,x) - \tau \sup_{k \geq 1} \varphi(k,x) = h(x) - h(\tau x) + g(x). \tag{14}$$

Iterating equation (14), we obtain $\varphi(n,x) = h(x) - h(\tau^n x) + \sum_0^{n-1} g(\tau^j x)$.

By claim 2 in Proposition 1, the cocycle (φ_n) is recurrent. Therefore, for a.e. x there is an infinite sequence $n_k = n_k(x)$ such that $\varphi(n_k, x)$ and $h(\tau^{n_k} x)$ are bounded. Since, by equation (13), $g(x) \leq 0$, the above formula implies that the series $\sum_{j=0}^{\infty} g(\tau^j x)$ converges for a.e. x. By the recurrence of the cocycle, it implies $g = 0$ a.e. Hence, by equation (14), φ is a coboundary, contrary to our assumption. Assuming that the condition $\inf_n \varphi_n(x) = -\infty$ for a. e. $x \in X$ is not satisfied, we derive that φ is a coboundary in a similar fashion. ∎

Combining Proposition 1 and Proposition 5 with the statements on ergodicity in [19] and [23], we strengthen the preceding results. Below we formulate the strengthened versions of Theorem 3 and Theorem 4. We leave the analogous strengthenings of Proposition 3 and Proposition 4 to the reader.

Corollary 5. *Let $\mathcal{C}$ be a closed set of polygons inside the unit square. For $O \in \mathcal{C}$ let $\tilde{P}_O$ be the corresponding periodic band with obstacles.*

The set of $O \in \mathcal{C}$ such that the geodesic flow on $\tilde{P}_O$ is zero-recurrent and has unbounded oscillations, is residual.

Corollary 6. *Let O be a polygon inside the unit square; let $\tilde{P}_O$ be the corresponding periodic band.*

Suppose that each π-irrational angle of O and each π-irrational angle between O and the horizontal axis admits the Vorobets approximation. Suppose, moreover, that not all of these angles are π-rational. Then the geodesic flow on $\tilde{P}_O$ is zero-recurrent, with unbounded oscillations.

We point out that there is a considerable interest in the physics literature in the conservativeness and related properties for $\mathbb{Z}$-periodic billiards. See, for instance, [3], [8], and the references there.

3. $\mathbb{Z}^2$-periodic polygonal surfaces: Dichotomies

Let R_0 be the unit square; let $O \subset \text{interior}(R_0)$ be a polygon. Set $\tilde{P}_O = \mathbb{R}^2 \setminus \cup_{(p,q)\in\mathbb{Z}^2}(O + (p,q))$. Thus, the $\mathbb{Z}^2$-periodic polygonal surface $\tilde{P}_O$ is the euclidean plane with a doubly periodic configuration of obstacles. It is the $\mathbb{Z}^2$-version of the band with obstacles studied in section 2.2 and section 2.3. The space $\tilde{P}_O$ may be called the *polygonal Lorenz gas*. When O is a rectangle, $\tilde{P}_O$ is the wind-tree model. See [9] and [16].

The study of geodesic flows for $\mathbb{Z}^2$-periodic polygonal surfaces is less complete that the corresponding study for $\mathbb{Z}$-periodic polygonal surfaces. In section 4 we will study in detail the directional flows for special directions in the wind-tree model.

In this section we expose a few general results on the conservativeness of arbitrary $\mathbb{Z}^2$-periodic polygonal surfaces $\tilde{P}_O$.

For concreteness of exposition, we will consider the surfaces $\tilde{P}_O$ when O is a triangle. We may then call $\tilde{P}_O$ a *triangular Lorenz gas*. The reader will easily extend the results that follow to arbitrary polygons in the unit square. We denote by $\mathcal{T}$ the topological space of triangles inside the unit square.

Proposition 6. *There is a dense G_δ set $\mathcal{D} \subset \mathcal{T}$ of triangles such that for $O \in \mathcal{D}$ the following dichotomy holds: i) the billiard in $\tilde{P}_O$ is zero-recurrent or ii) the billiard in $\tilde{P}_O$ is transient and satisfies equation* (6)*, with $d = 2, c > 0$.*

Proof. The billiard in $\tilde{P}_O$ fits into the framework of sections 1.2, 1.3. The compact polygonal surface $P_O = \tilde{P}_O/\mathbb{Z}^2$ is the standard torus with

a triangular obstacle. The flow $(U\tilde{P}_O, \tilde{T}^t, \tilde{\mu})$ is a skew product over the geodesic flow (UP_O, T^t, μ) with the fibre $\mathbb{Z}^2$. Set $\tilde{O} = \cup_{(p,q)\in\mathbb{Z}^2}(O + (p, q))$. The boundaries $\partial O, \partial\tilde{O}$ yield canonical cross-sections for the respective flows. Let $(U_{\partial O}P, \tau, \nu)$ and $(U_{\partial\tilde{O}}\tilde{P}, \tilde{\tau}, \tilde{\nu})$ be the respective billiard maps. Then $(U_{\partial\tilde{O}}\tilde{P}, \tilde{\tau}, \tilde{\nu})$ is a skew product over $(U_{\partial O}P, \tau, \nu)$; let $\varphi : U_{\partial O}P \to \mathbb{Z}^2$ be the corresponding displacement function. The flow $(U\tilde{P}_O, \tilde{T}^t, \tilde{\mu})$ is zero-recurrent (resp. transient) iff the map $(U_{\partial\tilde{O}}\tilde{P}, \tilde{\tau}, \tilde{\nu})$ is zero-recurrent (resp. transient).

For $(p, q) \in \mathbb{N}^2$ set $P_{(p,q)} = \tilde{P}_O/(p\mathbb{Z} \times q\mathbb{Z})$. Then $P_{(p,q)}$ are compact polygonal surfaces and $P_{(1,1)} = P_O$. If $(p, q), (p', q') \in \mathbb{N}^2$ are such that p divides p' and q divides q', then there is a finite covering $\pi^{(p',q')}_{(p,q)} : P_{(p',q')} \to P_{(p,q)}$. The set of $O \in \mathcal{T}$ such that the billiard map $(U_{\partial O}P_{(p,q)}, \tau_{(p,q)}, \nu_{(p,q)})$ is ergodic contains a dense G_δ set $\mathcal{D}_{(p,q)}$ [19].

For $O \in \cup_{(p>1,q)}\cup_{(p,q>1)}\mathcal{D}_{(p,q)}$, which is a dense G_δ, the map $(U_{\partial\tilde{O}}\tilde{P}, \tilde{\tau}, \tilde{\nu})$ satisfies the conditions of Proposition 2. ∎

Let now $O \subset R_0$ be a rational triangle; thus, the polygonal surface P_O is rational; let R_N be the corresponding dihedral group. For $\theta \in [0, \pi/N]$ let $(UP_\theta, T^t_\theta, \mu_\theta)$ and $(U\tilde{P}_\theta, \tilde{T}^t_\theta, \tilde{\mu}_\theta)$ be the directional geodesic flows on P_O and $\tilde{P}_O$ respectively. Let τ_θ and $\tilde{\tau}_\theta$ be the respective directional billiard maps. See section 1.3.

Proposition 7. *Let $O \subset R_0$ be a rational triangle such that $N = N(O)$ is even. Then for a.e. $\theta \in [0, \pi/N]$ the following dichotomy holds: i) The map $\tilde{\tau}_\theta$ is zero-recurrent or ii) the map $\tilde{\tau}_\theta$ is transient and satisfies equation* (6)*, with $d = 2, c > 0$.*

Proof. Let $S, \tilde{S}$ be the translation surface corresponding to $P_O, \tilde{P}_O$ respectively [10]. Then $\tilde{S}$ is a $\mathbb{Z}^2$-periodic translation surface, and $S = \tilde{S}/\mathbb{Z}^2$. We view $T^t_\theta, \tilde{T}^t_\theta$ as flows on $S_O, \tilde{S}_O$ respectively.

For $(p, q) \in \mathbb{N}^2$ set $S_{(p,q)} = \tilde{S}/(p\mathbb{Z} \times q\mathbb{Z})$. If $(p, q), (p', q') \in \mathbb{N}^2$ are such that p divides p' and q divides q', then there is a finite covering $g^{(p',q')}_{(p,q)} : S_{(p',q')} \to S_{(p,q)}$. We denote by $(S_{(p,q)}, T^t_{(p,q)}, \mu_{(p,q)})$ the flow in direction θ for the surface $S_{(p,q)}$. These flows are compatible with the coverings $g^{(p',q')}_{(p,q)}$; they form a projective family.

Let $E \subset [0, \pi/N]$ be the set of directions θ such that all directional flows $T^t_{(p,q)}$ are ergodic. For $\theta \in E$, by Lemma 3, ∂P_O yields cross-sections for the flows $(S_{(p,q)}, T^t_{(p,q)}, \mu_{(p,q)})$. Let $\tau_{(p,q)}$ be the Poincaré maps; let $\tilde{\tau}$ be the Poincaré map with respect to the corresponding cross-section for the flow $(\tilde{S}, \tilde{T}^t_\theta, \tilde{\mu})$. Then $\tilde{\tau}$ is a skew product over $\tau_{(p,q)}$.

By Lemma 5, the corresponding displacement functions are centered. Proposition 2 implies the claim for $\tilde{\tau}$.

It remains to show that the set E has full measure. For each surface $S_{(p,q)}$ the set $E_{(p,q)}$ of uniquely ergodic directions has full measure [19]. Since $E = \cap_{(p,q)\in\mathbb{N}^2} E_{(p,q)}$, the claim follows. ∎

4. $\mathbb{Z}^2$-periodic polygonal surfaces: Rectangular Lorenz gas

We will study the polygonal Lorenz gas $\tilde{P}_O$ of section 3 when O is a rectangle: The rectangular Lorenz gas. In the physics literature this is known as the *wind-tree model* [16]; it is of some interest for foundations of statistical physics [9]. We begin by introducing notation. Let $0 < a, b < 1$. For $(m,n) \in \mathbb{Z}^2$ let $R_{(m,n)}(a,b) \subset \mathbb{R}^2$ be the $a \times b$ rectangle centered at (m,n) whose sides are parallel to the coordinate axes. The Lorenz gas with rectangular obstacles of size $a \times b$ corresponds to the polygonal surface

$$\tilde{P}(a,b) = \mathbb{R}^2 \setminus \bigcup_{(m,n)\in\mathbb{Z}^2} R_{(m,n)}(a,b). \tag{15}$$

The quotient surface $P(a,b) = \tilde{P}(a,b)/\mathbb{Z}^2$ is the unit torus with a rectangular hole.[7]

We will modify the notation of section 1 as follows. The surface $P(a,b)$ is rational and its dihedral group is R_2. We identify U/R_2 with $[0,\pi/2]$. We will suppress (a,b) from our notation whenever this does not cause confusion. For $\theta \in [0,\pi/2]$ we denote by $(\tilde{Z}_\theta, \tilde{T}^t_\theta, \tilde{\mu}_\theta)$ and $(Z_\theta, T^t_\theta, \mu_\theta)$ (resp. $(\tilde{X}_\theta, \tilde{\tau}_\theta, \tilde{\nu}_\theta)$ and $(X_\theta, \tau_\theta, \nu_\theta)$) the billiard flow (resp. billiard map) for $\tilde{P}(a,b)$ and $P(a,b)$ respectively.

4.1. **Rational directions and small obstacles.**

A direction $\theta \in [0,\pi/2]$ is rational if $\tan\theta \in \mathbb{Q}$. Rational directions $\theta(p,q) = arctan(q/p)$ correspond to pairs $(p,q) \in \mathbb{N}^2$ with relatively prime p,q. When there is no danger of confusion, we will use the notation (p,q) instead of $\theta(p,q)$.

Let $R(a,b) = ABCD$ be the rectangle in the unit torus. Let $\theta \in (0,\pi/2)$. The space X_θ consists of unit vectors pointing outward, whose base points belong to $ABCD$ and whose directions belong to the set $\{\pm\theta, \pi \pm \theta\}$. See figure 6.

[7] The polygonal surface $\tilde{P}(0,b)$ or $\tilde{P}(a,0)$ is the euclidean plane with a $\mathbb{Z}^2$-periodic family of barriers. The billiard flow is then transient. Hence, we assume that $a, b > 0$.

We say that the Lorenz gas $\tilde{P}(a,b)$ has *small obstacles with respect to* (p,q) if the geodesics in $P(a,b)$ emanating from A or C in the direction $\theta(p,q)$ return to either point without encountering $R(a,b)$ on the way.

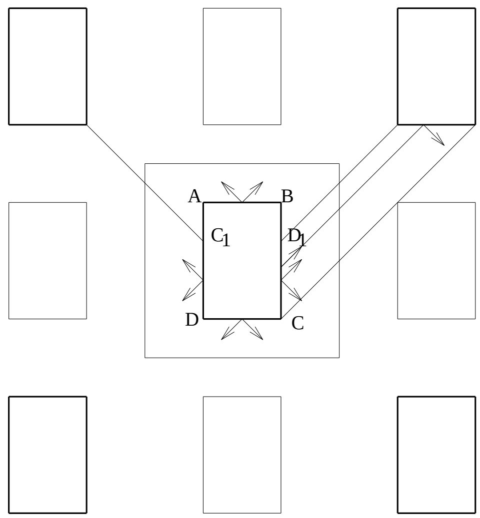

FIGURE 6. *The cross-section for the conservative part of the billiard flow in direction $\pi/4$.*

Lemma 6. *The small obstacles condition is satisfied iff*

$$qa + pb \leq 1. \tag{16}$$

The inequality in equation (16) *is strict iff the directional geodesic flow* $(\tilde{Z}_{(p,q)}, \tilde{T}^t_{(p,q)}, \tilde{\mu}_{(p,q)})$ *has a set of positive measure of orbits that do not encounter obstacles.*

Proof. The condition is satisfied iff $R(a,b)$ fits between two parallel lines with slopes q/p and vertical displacement $1/p$. By an elementary calculation, this is possible iff

$$a\frac{q}{p} + b \leq \frac{1}{p}.$$

Moreover, the equality in equation (16) holds iff $R(a,b)$ takes all of the space between the boundary components of the strip. See figure 7. ∎

In what follows we fix (p,q) and assume that the inequality equation (16) is satisfied. We identify $X_{(p,q)}$ with 2 copies of the rectangle $ABCD$; the copy denoted by $X_+ = (ABCD)_+$ (resp. $X_- = (ABCD)_-$) carries the outward pointing vectors in the directions $\theta, \pi+\theta$ (resp. $\pi-\theta, 2\pi-\theta$). Figure 8 illustrates this. We will now investigate

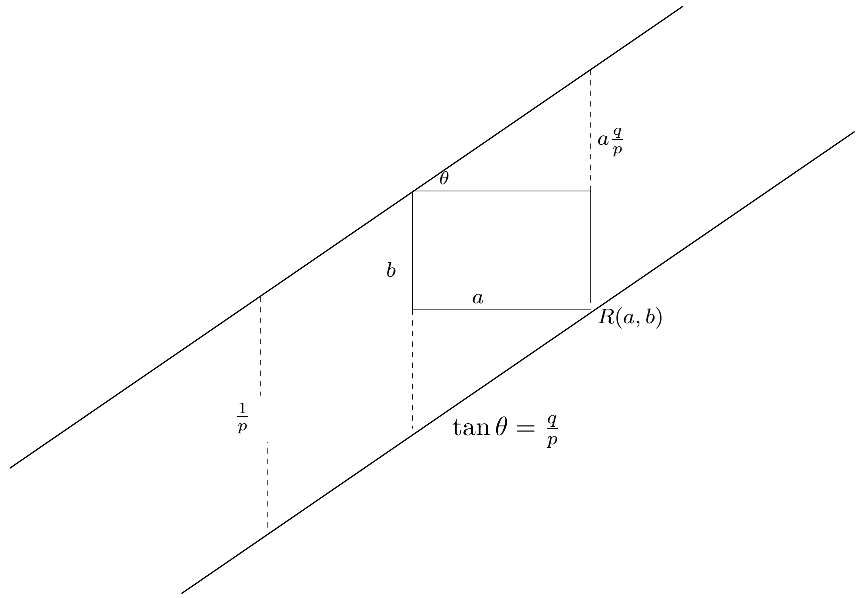

FIGURE 7. *The smallness of obstacles condition: Fitting the rectangle in a strip.*

the Poincaré map $\tau_{(p,q)} : X_{(p,q)} \to X_{(p,q)}$; we suppress the subscripts when it causes no confusion.

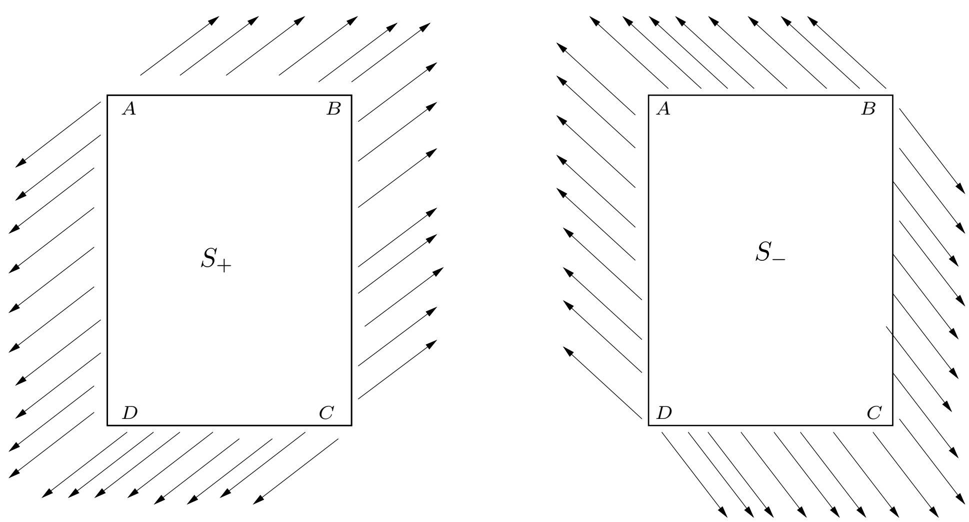

FIGURE 8. *The cross-section $X_{(p,q)}$ and the two corresponding rectangles.*

Lemma 7. 1. *There are natural identifications of X_+ and X_- with the circle $\mathbb{R}/\mathbb{Z}$ endowed with distinguished points A, B, C, D; their relative positions are given by*

$$|AB| = |CD| = \frac{qa}{2(qa+pb)}, \ |BC| = |DA| = \frac{pb}{2(qa+pb)}. \tag{17}$$

2. *Set $\tau_\pm = \tau|_{X_\pm}$. Then $\tau_+ : X_+ \to X_-,\ \tau_- : X_- \to X_+$. Set $S = \mathbb{R}/\mathbb{Z}$. With the identifications $X_\pm = S$, the maps $\tau_+ : S \to S$ and $\tau_- : S \to S$ are the orthogonal reflections about the axes AC and BD respectively. The maps $\tau_-\tau_+ : X_+ \to X_+$ and $\tau_+\tau_- : X_- \to X_-$ are the rotations of S by $\dfrac{qa}{(qa+pb)}$ and $\dfrac{pb}{(qa+pb)}$ respectively.*

Proof. Vectors emanating from the rectangular obstacle in direction η at the first return assume the direction $r(\eta)$, where r is a reflection in R_2. Thus, $\tau : X_+ \to X_-,\ X_- \to X_+$.

Let s and γ be the arclength and the angle coordinates on the billiard cross-section. Up to a constant factor, the invariant measure for the billiard map has density $d\nu = \sin\gamma\, ds\, d\gamma$.[8] Integrating, we have $\nu(AB) = \nu(CD) = qa, \nu(BC) = \nu(DA) = pb$, up to a constant factor. Normalizing $\nu(S) = 1$, we obtain equation (17).

By construction, the maps $\tau_\pm : S \to S$ are orientation reversing diffeomorphisms. Since they preserve the arclength, they are isometries. Thus, $\tau_\pm : S \to S$ are orthogonal reflections. By construction, τ_+ (resp. τ_-) fixes the points A, C (resp. B, D). These pairs of points correspond to the axes of reflections when we identify S with the unit circle in $\mathbb{R}^2$.

■

In what follows we will sometimes view $\tau_\pm$ as isometries of the unit circle, $\tau_\pm : S \to S$, and sometimes as mappings between the two copies of the circle, $\tau_+ : S_+ \to S_-,\ \tau_- : S_- \to S_+$.

Set $Z_\pm = S_\pm \times \mathbb{Z}^2$ and $Z = Z_+ \cup Z_-$. We set $\tilde\tau = \tilde\tau_{(p,q)}$. Then $\tilde\tau : Z \to Z$ is the Poincaré map; it interchanges the sets Z_+, Z_-. We use the notation $\tilde\tau_\pm(x, g) = (\tau_\pm(x), g + \varphi_\pm(x))$. Thus, $\varphi_\pm : S \to \mathbb{Z}^2$ are the displacement functions. The following is immediate from Lemma 7 and figure 9.

Lemma 8. *The displacement functions $\varphi_\pm : S \to \mathbb{Z}^2$ are constant on the circular arcs ABC, CDA, DAB, BCD. We have*

$$\varphi_+|_{ABC} = (p, q),\ \varphi_+|_{CDA} = (-p, -q),\ \varphi_-|_{DAB} = (-p, q),\ \varphi_-|_{BCD} = (p, -q).$$

[8]See, e.g., [15] for this material.

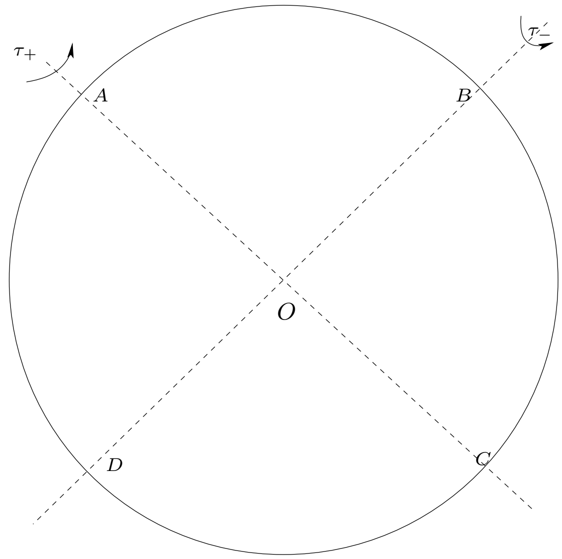

FIGURE 9. *The circle and the two orthogonal reflections.*

Set

$$A_1 = \tau_-^{-1}(A), B_1 = \tau_+^{-1}(B), C_1 = \tau_-^{-1}(C), D_1 = \tau_+^{-1}(D).$$

Our next result describes the transformation $\tilde{\tau}^2 : Z \to Z$. We set $\tilde{\tau}^2_\pm = \tilde{\tau}^2|_{Z_\pm}$. Recall that we have identified S and $\mathbb{R}/\mathbb{Z}$. We will usually denote by $x+y$ the operation in $\mathbb{R}/\mathbb{Z}$. If the danger of confusion arises, we will write $x+y \mod 1$.

Proposition 8. *We have*

$$\begin{aligned}(\tilde{\tau}^2)_+(x,g) &= (x+\frac{qa}{qa+pb}, g+\psi_+(x)),\\ (\tilde{\tau}^2)_-(x,g) &= (x+\frac{pb}{qa+pb}, g+\psi_-(x)).\end{aligned}$$

The displacement functions $\psi_\pm$ take values $(\pm 2p, 0), (0, \pm 2q)$. Each $\psi_\pm$ determines a partition of S into four intervals such that $\psi_\pm = const$ on each interval. The endpoints of these intervals belong to the set $A, B, C, D, A_1, B_1, C_1, D_1$.

Proof. We have $(\tau^2)_+ = \tau_-\tau_+, (\tau^2)_- = \tau_+\tau_-$. The product of two orthogonal reflections is the rotation by twice the angle between their axes. The values of angles follow from Lemma 7.

We have $\psi_\pm(x) = \varphi_\pm(x) + \varphi_\mp(\tau_\pm(x))$. By Lemma 8, $\psi_\pm$ are constant on the circular arcs which are the intersections of half-circles

ABC, CDA, DAB, BCD with half-circles $A_1BC_1, C_1DA_1, D_1AB_1, B_1CD_1$. We have

$$\psi_+|_{ABC\cap D_1AB_1} = \varphi_+|_{ABC} + \varphi_-|_{DAB} = (p,q) + (-p,q) = (0,2q);$$

$$\psi_+|_{ABC\cap B_1CD_1} = \varphi_+|_{ABC} + \varphi_-|_{BCD} = (p,q) + (p,-q) = (2p,0);$$

$$\psi_+|_{CDA\cap D_1AB_1} = \varphi_+|_{CDA} + \varphi_-|_{DAB} = (-p,-q) + (-p,q) = (-2p,0);$$

$$\psi_+|_{CDA\cap B_1CD_1} = \varphi_+|_{CDA} + \varphi_-|_{BCD} = (-p,-q) + (p,-q) = (0,-2q).$$

Analogously

$$\psi_-|_{DAB\cap A_1BC_1} = \varphi_-|_{DAB} + \varphi_+|_{ABC} = (-p,q) + (p,q) = (0,2q);$$

$$\psi_-|_{DAB\cap C_1DA_1} = \varphi_-|_{DAB} + \varphi_+|_{CDA} = (-p,q) + (-p,-q) = (-2p,0);$$

$$\psi_-|_{BCD\cap A_1BC_1} = \varphi_-|_{BCD} + \varphi_+|_{ABC} = (p,-q) + (p,q) = (2p,0);$$

$$\psi_-|_{BCD\cap C_1DA_1} = \varphi_-|_{BCD} + \varphi_+|_{CDA} = (p,-q) + (-p,-q) = (0,-2q).$$

∎

We set

$$\alpha = \frac{qa}{qa+pb},\ \beta = \frac{pb}{qa+pb}. \tag{18}$$

Then $0 < \alpha, \beta < 1$ and $\alpha + \beta = 1$. In what follows we assume that $\alpha < \beta$ or, equivalently, $qa < pb$. This assumption allows us to avoid extra computations. The case $\beta < \alpha$ reduces to this by switching the coordinate axes. We identify S_+ (resp. S_-) with $[0,1]$ so that the points A, B, C, D (resp. D, A, B, C) go to $0, \frac{\alpha}{2}, \frac{1}{2}, \frac{1}{2} + \frac{\alpha}{2}$ (resp. $0, \frac{1}{2} - \frac{\alpha}{2}, \frac{1}{2}, 1 - \frac{\alpha}{2}$) respectively. With these identifications, $\psi_\pm : S \to \mathbb{Z}^2$ are piecewise constant functions on $[0,1]$. We will now explicitly describe them. The formulas below follow from Proposition 8 by straightforward calculations; we leave them to the reader.

Proposition 9. *The function $\psi_+ : [0,1] \to \mathbb{Z}^2$ is given by*

$$\psi_+(x) = \begin{cases} (0,2q) & \text{on }]0, \frac{1}{2} - \frac{\alpha}{2}[, \\ (2p,0) & \text{on }]\frac{1}{2} - \frac{\alpha}{2}, \frac{1}{2}[, \\ (0,-2q) & \text{on }]\frac{1}{2}, 1 - \frac{\alpha}{2}[, \\ (-2p,0) & \text{on }]1 - \frac{\alpha}{2}, 1[. \end{cases} \tag{19}$$

The function $\psi_- : [0,1] \to \mathbb{Z}^2$ is given by

$$\psi_-(x) = \begin{cases} (-2p,0) & \text{on }]0, \frac{\alpha}{2}[, \\ (0,2q) & \text{on }]\frac{\alpha}{2}, \frac{1}{2}[, \\ (2p,0) & \text{on }]\frac{1}{2}, \frac{1}{2} + \frac{\alpha}{2}[, \\ (0,-2q) & \text{on }]\frac{1}{2} + \frac{\alpha}{2}, 1[. \end{cases} \tag{20}$$

The following properties of $\psi_\pm : [0,1] \to \mathbb{Z}^2$ are immediate from equations (19), (20). The lengths of intervals of continuity are $\frac{\alpha}{2}, \frac{\beta}{2}$, and they alternate. Each function takes four values which generate the subgroup $H_{(p,q)} = 2p\mathbb{Z} \oplus 2q\mathbb{Z} \subset \mathbb{Z}^2$. Using the isomorphism $(a,b) \mapsto (2pa, 2qb)$ of $\mathbb{Z}^2$ and $H_{(p,q)}$, we replace the displacement functions ψ_+ and ψ_- by piecewise constant functions on $[0,1]$ that do not depend on p,q. Let Ψ be the function corresponding to ψ_+. Then

$$\Psi(x) = \begin{cases} (0,1) & \text{on }]0, \frac{1}{2} - \frac{\alpha}{2}[, \\ (1,0) & \text{on }]\frac{1}{2} - \frac{\alpha}{2}, \frac{1}{2}[, \\ (0,-1) & \text{on }]\frac{1}{2}, 1 - \frac{\alpha}{2}[, \\ (-1,0) & \text{on }]1 - \frac{\alpha}{2}, 1[. \end{cases} \tag{21}$$

4.2. **Ergodic decompositions for the billiard dynamics.**
Let $\tilde{\tau} : X \times \mathbb{Z}^2 \to X \times \mathbb{Z}^2$ be the billiard map in direction (p,q) for the Lorenz gas with rectangular obstacles of size $a \times b$. Recall that we have identified X with 2 copies of the unit circle: $X = S_+ \cup S_-$. Let $G_{(p,q)} \subset \mathbb{Z}^2$ be the group generated by (p,q) and $(p,-q)$. Then $|\mathbb{Z}^2/G_{(p,q)}| = 2pq$. If G is any countable group, we will denote by $\tilde{\nu}$ the measure on $X \times G$ which is the product of the Lebesgue measure on X and the counting measure on G.

Theorem 5. *Let $(p,q) \in \mathbb{N}^2$ with p,q relatively prime. Let $a,b > 0$ satisfy $qa + pb \leq 1$; suppose that a/b is irrational. For $\bar{g} \in \mathbb{Z}^2/G_{(p,q)}$ denote by $\bar{g} + G_{(p,q)} \subset \mathbb{Z}^2$ the corresponding cosets. Then the following holds.*
1. *For $\bar{g} \in \mathbb{Z}^2/G_{(p,q)}$ the sets $X \times (\bar{g} + G_{(p,q)}) \subset X \times \mathbb{Z}^2$ are $\tilde{\tau}$-invariant. The dynamical systems $(X \times (\bar{g} + G_{(p,q)}), \tilde{\tau}, \tilde{\nu})$ are ergodic; they are isomorphic for all $\bar{g} \in \mathbb{Z}^2/G_{(p,q)}$.*
2. *The partition*

$$X \times \mathbb{Z}^2 = \cup_{\bar{g} \in \mathbb{Z}^2/G_{(p,q)}} X \times (\bar{g} + G_{(p,q)}). \tag{22}$$

yields the decomposition of the dynamical system $(X \times \mathbb{Z}^2, \tilde{\tau}, \tilde{\nu})$ into $2pq$ isomorphic ergodic components.

Proof. Set $\bar{G} = \mathbb{Z}^2/G_{(p,q)}$. Lemma 7 and Lemma 8 identify $\tilde{\tau}$ with the collection of transformations $\tilde{\tau}|_{\bar{g}} : S_\pm \times (\bar{g} + G_{(p,q)}) \to S_\mp \times (\bar{g} + G_{(p,q)})$, where $\bar{g} \in \bar{G}$. This implies the first part of claim 1.

Propositions 8 and 9 represent the restrictions $\tilde{\tau}^2_\pm|_{\bar{g}} : S_\pm \times (\bar{g} + G_{(p,q)}) \to S_\pm \times (\bar{g} + G_{(p,q)})$ as skew product transformations $\rho_{\sigma,\psi}$ over certain rotations $s \mapsto s + \sigma$ on $S = \mathbb{R}/\mathbb{Z}$ with particular displacement functions ψ. They do not depend on $\bar{g} \in \bar{G}$. This proves the third part of claim 1.

With the notation of equations (18), (19), (20), we have

$$\tilde{\tau}_+^2|_{\bar{g}} = \rho_{\alpha,\psi_+},\ \tilde{\tau}_-^2|_{\bar{g}} = \rho_{\beta,\psi_-}. \tag{23}$$

Since $\tilde{\tau}$ interchanges $S_+ \times (\bar{g} + G_{(p,q)})$ and $S_- \times (\bar{g} + G_{(p,q)})$, the ergodicity of $\tilde{\tau}|_{\bar{g}}$ would follow from the ergodicity of skew products $\rho_{\alpha,\psi_+}, \rho_{\beta,\psi_-}$. By symmetry, it suffices to prove the ergodicity of ρ_{α,ψ_+}. Let $\Psi : S \to \mathbb{Z}^2$ be given by equation (21), and let $\rho_{\alpha,\Psi} : S \times \mathbb{Z}^2 \to S \times \mathbb{Z}^2$ be the corresponding skew product. The isomorphism $G_{(p,q)} = \mathbb{Z}^2$ and equation (21) yield $\rho_{\alpha,\psi_+} = \rho_{\alpha,\Psi}$. By Theorem 9 in section 5.3, $\rho_{\alpha,\Psi}$ is ergodic for any irrational α. We have established claim 1. Claim 2 is immediate from it. ■

Theorem 5 describes the ergodic decomposition of the billiard map in direction (p,q) on the polygonal surface $\tilde{P}(a,b)$. We will now describe the decomposition of the geodesic flow in direction (p,q).[9] The configuration space for the directional flow $(\tilde{Z}, \tilde{T}^t_{(p,q)}, \tilde{\mu})$ consists of unit vectors in directions $(\pm p, \pm q)$ with base points in $\tilde{P}(a,b)$. For $\tilde{z} \in \tilde{Z}$ we denote by $\gamma(\tilde{z}) \subset \tilde{P}(a,b)$ the geodesic it generates. Let $\tilde{C} \subset \tilde{Z}$ (resp. $\tilde{D} \subset \tilde{Z}$) be the set of $\tilde{z} \in \tilde{Z}$ such that $\gamma(\tilde{z})$ encounters (resp. does not encounter) rectangular obstacles. Then $\tilde{Z} = \tilde{C} \cup \tilde{D}$, a disjoint union. For $\bar{g} \in \bar{G}$ set $\tilde{O}(\bar{g}) = \cup_{(m,n)\in(\bar{g}+G_{(p,q)})} R_{(m,n)}(a,b)$. Thus, $\tilde{O}(\bar{g})$ is the union of obstacles $R_{(m,n)}(a,b)$, as (m,n) varies in the coset $\bar{g} + G_{(p,q)}$. Let $\tilde{C}(\bar{g}) \subset \tilde{C}$ be the set of phase points $\tilde{z} \in \tilde{Z}$ such that $\gamma(\tilde{z})$ encounters obstacles in $\tilde{O}(\bar{g})$. Let $\tilde{\mu}_{\bar{g}}$ be the restriction of $\tilde{\mu}$ to $\tilde{C}(\bar{g})$.

Theorem 6. *Let $(p,q) \in \mathbb{N}^2$ with p,q relatively prime. Let $a,b > 0$ satisfying equation* (16) *be such that a/b is irrational. Let $(\tilde{Z}, \tilde{T}^t_{(p,q)}, \tilde{\mu})$ be the directional flow. Then the following holds.*
1. *The sets $\tilde{C}(\bar{g}), \bar{g} \in \bar{G}$, are $\tilde{T}^t$-invariant; the dynamical systems $(\tilde{C}(\bar{g}), \tilde{T}^t, \tilde{\mu}_{\bar{g}})$ are ergodic and pairwise isomorphic. The partition $\tilde{C} = \cup_{\bar{g}\in\bar{G}}\tilde{C}(\bar{g})$ yields the ergodic decomposition*

$$(\tilde{C}, \tilde{T}^t_{(p,q)}, \tilde{\mu}) = \cup_{\bar{g}\in\bar{G}}(\tilde{C}(\bar{g}), \tilde{T}^t_{(p,q)}, \tilde{\mu}_{\bar{g}}) \tag{24}$$

of the conservative part of the flow $(\tilde{Z}, \tilde{T}^t_{(p,q)}, \tilde{\mu})$.
2. *The dissipative part $\tilde{D}$ is trivial iff we have equality in equation* (16). *Suppose that the inequality in equation* (16) *holds. Then $\tilde{D} = L \times \mathbb{R}$, where L is a countable union of disjoint intervals of the same length.*

[9] To simplify notation, we will suppress the dependence on (p,q) whenever this does not cause confusion.

The restriction of μ to $\tilde{D}$ is the product of lebesgue measures on L and $\mathbb{R}$; the flow $(L \times \mathbb{R}, \tilde{T}^t_{(p,q)}, \tilde{\mu})$ is the translation flow along $\mathbb{R}$.

Proof. By definition, the restriction of the flow $\tilde{T}^t_{(p,q)}$ to $\tilde{D}$ is dissipative. Claim 2 is immediate from Lemma 6.

The flow $(\tilde{C}, \tilde{T}^t_{(p,q)}, \tilde{\mu})$ is a suspension flow over the transformation $(X \times \mathbb{Z}^2, \tilde{\tau}, \tilde{\nu})$. Claim 1 now follows directly from Theorem 5. In particular, equation (24) follows from the ergodic decomposition of $(X \times \mathbb{Z}^2, \tilde{\tau}, \tilde{\nu})$ given by equation (22). ∎

The proposition below relates the ergodic decomposition equation (24) to an equidistribution of billiard orbits. It holds under the assumptions of Theorem 6.

The geodesic $\gamma(\tilde{z})$ generated by $\tilde{z} \in \tilde{Z}$ is a curve in the polygonal surface $\tilde{P}(a,b)$. We will use the notation $\gamma_{\tilde{z}}(t), 0 \leq t$, for this curve, parameterized by the arclength. For $(m,n) \in \mathbb{Z}^2, \tilde{z} \in \tilde{C}$ and $T > 0$ let $N(\tilde{z}, T; (m,n))$ be the number of times $0 \leq t \leq T$ such that the billiard orbit $\gamma_{\tilde{z}}(t)$ encounters the obstacle $R_{(m,n)}(a,b)$.

Proposition 10. *Let $(m,n), (m',n') \in \mathbb{Z}^2$. Then the following dichotomy holds.*
1. Suppose that the numbers $\frac{m-m'}{p}, \frac{n-n'}{q}$ are integers of the same parity. Then there is a $\tilde{T}^t$ invariant subset $\tilde{E} \subset \tilde{C}$ of infinite measure determined by the coset $(m,n) + G_{(p,q)}$, and such that for $\tilde{\mu}$-almost every $\tilde{z} \in \tilde{E}$ both functions $N(\tilde{z}, T; (m,n)), N(\tilde{z}, T; (m',n'))$ go to infinity as $T \to \infty$. Moreover, for $\tilde{\mu}$-almost every $\tilde{z} \in \tilde{E}$ we have

$$\lim_{T \to \infty} \frac{N(\tilde{z}, T; (m,n))}{N(\tilde{z}, T; (m',n'))} = 1 \tag{25}$$

The set $\tilde{C} \setminus \tilde{E}$ also has infinite measure. For $\tilde{\mu}$-almost every $\tilde{z} \in \tilde{C} \setminus \tilde{E}$ we have

$$N(\tilde{z}, T; (m,n)) = N(\tilde{z}, T; (m',n')) = 0.$$

2. Suppose that the above assumption on $(m,n), (m',n')$ is not satisfied. Then for $\tilde{\mu}$-almost every $\tilde{z} \in \tilde{C}$ one of the following possibilities holds:

a) $N(\tilde{z}, T; (m,n)) = N(\tilde{z}, T; (m',n')) = 0$;
b) $N(\tilde{z}, T; (m,n)) = 0$, $N(\tilde{z}, T; (m',n')) \to \infty$;
c) $N(\tilde{z}, T; (m',n')) = 0$, $N(\tilde{z}, T; (m,n)) \to \infty$.

Proof. 1. Recall that $(\tilde{C}, \tilde{T}^t_{(p,q)}, \tilde{\mu})$ is a suspension flow over the billiard map $(X \times \mathbb{Z}^2, \tilde{\tau}, \tilde{\nu})$. The set X consists of unit vectors with directions

$(\pm p, \pm q)$ based on the boundary of the rectangle $R(a,b)$. Let $\tilde{X} = X \times \mathbb{Z}^2$ and for $(m,n) \in \mathbb{Z}^2$ set $\tilde{X}_{(m,n)} = X \times \{(m,n)\}$. Then

$$\tilde{X} = \cup_{(m,n)\in\mathbb{Z}^2} \tilde{X}_{(m,n)},$$

a disjoint union. We will use the ergodic decomposition of $(X \times \mathbb{Z}^2, \tilde{\tau}, \tilde{\nu})$ established in Thorem 5. For $\bar{g} \in \bar{G}$ let $\tilde{X}(\bar{g}) \subset \tilde{X}$ be the corresponding ergodic component of $(X \times \mathbb{Z}^2, \tilde{\tau}, \tilde{\nu})$. We denote by $(m,n) \mapsto \overline{(m,n)}$ the projection of $\mathbb{Z}^2$ onto $\bar{G} = \mathbb{Z}^2/G_{(p,q)}$. Then, by equation (22),

$$\tilde{X}(\bar{g}) = \cup_{\overline{(m,n)}=\bar{g}} \tilde{X}_{(m,n)}.$$

Denote by $1_{(m,n)}(\tilde{x})$ the function $1_{\tilde{X}_{(m,n)}} : \tilde{X} \to \mathbb{Z}$. For $k \in \mathbb{N}$ set

$$f(\tilde{x}, k; (m,n)) = \sum_{i=0}^{k} 1_{(m,n)}(\tilde{\tau}^i(\tilde{x})).$$

Our assumption on $(m,n), (m',n')$ is equivalent to $\overline{(m,n)} = \overline{(m',n')}$. Set $\overline{(m,n)} = \bar{g} \in \bar{G}$. Then $\tilde{X}_{(m,n)}, \tilde{X}_{(m',n')} \subset \tilde{X}(\bar{g})$. We use the ergodic theorem for dynamical systems with infinite invariant measure [1]. It states that for a.e. $\tilde{x} \in \tilde{X}(\bar{g})$

$$\lim_{k\to\infty} \frac{f(\tilde{x}, k; (m,n))}{f(\tilde{x}, k; (m',n'))} = \lim_{k\to\infty} \frac{\sum_{i=0}^{k} 1_{(m,n)}(\tilde{\tau}^i(\tilde{x}))}{\sum_{i=0}^{k} 1_{(m',n')}(\tilde{\tau}^i(\tilde{x}))} = \frac{\tilde{\nu}(\tilde{X}_{(m,n)})}{\tilde{\nu}(\tilde{X}_{(m',n')})}.$$

The volume $\tilde{\nu}(\tilde{X}_{(k,l)})$ does not depend on $(k,l) \in \mathbb{Z}^2$. We have $0 < \tilde{\nu}(\tilde{X}_{(k,l)}) < \infty$, and $\tilde{\nu}(\tilde{X}_{(k,l)})$ is determined by $a \times b$ and (p,q). See Lemma 7. Hence, the preceding equation implies the formula

$$\lim_{k\to\infty} \frac{f(\tilde{x}, k; (m,n))}{f(\tilde{x}, k; (m',n'))} = 1, \tag{26}$$

which holds, as usual, for a.e. $\tilde{x} \in \tilde{X}(\bar{g})$.

We will now outline an asymptotic relationship between the functions $N(\tilde{z}, T; (m,n))$ and $f(\tilde{x}, k; (m,n))$. Let $P = \tilde{P}/\mathbb{Z}^2$ be the compact polygonal surface; to simplify our notation, we suppress the dependence on $a \times b$ and on (p,q). Recall that P is the standard torus with a rectangular obstacle. Let (Z, T^t, μ) and (X, τ, ν) be the billiard flow and the billiard map in the direction (p,q) for P. Then (Z, T^t, μ) is a suspension flow over (X, τ, ν); the roof function $r(x) : X \to \mathbb{R}_+$ is the time it takes for the forward billiard orbit $\gamma_x(t), 0 < t$, to return to the cross-section. The mean time of return is given by

$$\bar{r} = \frac{\int_X r(x) d\nu}{\nu(X)}.$$

Let $A \subset X$ be a measurable set. We associate with A two functions. The function $N_A(z,T) : Z \times \mathbb{R}_+ \to \mathbb{N}$ is the number of times $0 < t < T$ the billiard flow orbit $\gamma_z(t)$ encounters A; the function $f_A(x,k) : X \times \mathbb{N} \to \mathbb{N}$ is the number of times $0 \le i < k$ the billiard map orbit $\tau^i(x)$ returns to A. Suppose that (X, τ, ν) is ergodic. Then for ν-a. e. $x \in X$ we have

$$N_A(x,T) = f_A(x, \lfloor \frac{T}{\bar{r}} \rfloor) + o(T). \tag{27}$$

We point out that equation (27) holds for ergodic suspension flows, in general. In our setting the roof function has a simple geometric meaning. Let $|\cdot|$ denote the euclidean norm on $\mathbb{R}^2$. Recall that $\varphi : X \to \mathbb{Z}^2 \subset \mathbb{R}^2$ is the displacement function. The elements $\tilde{x} \in \tilde{X}$ are vectors in $\mathbb{R}^2$ based at boundary points of the rectangles $R_{(m,n)}(a,b)$ as $(m,n) \in \mathbb{Z}^2$. Let $\tilde{x} \mapsto x$ be the projection of $\tilde{X}$ onto X. For $x \in X$ let $b(x) \in \mathbb{R}^2$ be the base point. Then $\tilde{x} \mapsto b(\tau(x)) - b(x) + \varphi(x)$ is a well defined mapping of $\tilde{X}$ to $\mathbb{R}^2$. We then have

$$r(\tilde{x}) = |b(\tilde{\tau}(x)) - b(x) + \varphi(x)|. \tag{28}$$

Note that the function $r : \tilde{X} \to \mathbb{R}_+$ is $\mathbb{Z}^2$-invariant. The mean return time to the cross-section $\tilde{X}$ is equal to the mean return time to the quotient cross-section X. We have

$$\bar{r} = \frac{\int_X |b(\tau(x)) - b(x) + \varphi(x)| d\nu(x)}{\nu(X)}. \tag{29}$$

Thus, $0 < \bar{r} < \infty$. Set $\tilde{E} = \tilde{C}(\bar{g})$. Combining equations (26), (27), (29), we obtain the former part of our claim. We have $\tilde{C} \setminus \tilde{E} = \cup_{\bar{h} \in \bar{G} \setminus \{\bar{g}\}} \tilde{C}(\bar{h})$. The remaining part of our claim follows from the preceding discussion and equation (27).

2. Set $\bar{g} = \overline{(m,n)}, \bar{g}' = \overline{(m',n')}$. The assumption on $(m,n),(m',n')$ does not hold iff $\bar{g} \neq \bar{g}'$. Thus, $\tilde{X}(\bar{g})$ and $\tilde{X}(\bar{g}')$ (resp. $\tilde{C}(\bar{g})$ and $\tilde{C}(\bar{g}')$) are distinct ergodic components of $\tilde{X}$ (resp. $\tilde{C}$). Equations a), b), c) follow from the preceding discussion. Equation a) holds when $\tilde{z} \in \tilde{C} \setminus (\tilde{C}(\bar{g}) \cup \tilde{C}(\bar{g}')$; equation b) (resp. c)) holds when $\tilde{z} \in \tilde{C}(\bar{g}')$ (resp. $\tilde{z} \in \tilde{C}(\bar{g})$). ∎

The following is immediate from Proposition 10.

Corollary 7. *Let $(m,n),(m',n') \in \mathbb{Z}^2$. Then for almost every $\tilde{z} \in \tilde{C}$ the ratio $N(\tilde{z},T;(m,n))/N(\tilde{z},T;(m',n'))$ converges to either 1, or 0, or infinity, as $T \to \infty$.*

5. Ergodicity of cocycles over irrational rotations

The subject of this section is the ergodic theory for a certain class of skew product transformations. The results are instrumental in obtaining ergodic decompositions for directional flows in the rectangular Lorenz gas model. See Theorem 5, Theorem 6, and Corollary 10.

Throughout this section, we will use the following setting. Let G be a locally compact abelian group.[10] Set $X = \mathbb{R}/\mathbb{Z}$. For $0 < \alpha < 1$ let $\rho_\alpha : X \to X$ be the rotation $x \mapsto x + \alpha \bmod 1$. Let $\Phi : X \to G$ be a piecewise constant function. Define the transformation $\rho_{\alpha,\Phi} : X \times G \to X \times G$ by $(x, g) \mapsto (\rho_\alpha(x), g + \Phi(x))$. Let Leb denote the Lebesgue measure on X; let μ be the measure on $X \times G$ which is the cartesian product of Leb and a Haar measure on G. The dynamical system $(X \times G, \rho_{\alpha,\Phi}, \mu)$ is the *skew product* over ρ_α, with the fibre G and the displacement function Φ. Let (Φ_n) be the cocycle corresponding to Φ and ρ_α. See Definition 1. If the dynamical system $(X \times G, \rho_{\alpha,\Phi}, \mu)$ is ergodic, we will say that *the cocycle* (Φ_n) *is ergodic*.

In the studies of ergodicity for $(X \times G, \rho_{\alpha,\Phi}, \mu)$ it is common to assume that the points of discontinuity of Φ are not arithmetically related to α. In view of equation (21), this assumption does not hold for our applications in section 4. Thus, we will expose two approaches to proving the ergodicity of cocycles over irrational rotations. One of them is based on the "well distributed discontinuities" property for a cocycle.[11] See Definition 6. The cocycles (Ψ_n), needed for our applications, satisfy (wdd) for generic α. This approach allows us to establish the ergodicity of (Ψ_n), and similar cocycles, for generic rotation angles. We develop this approach in section 5.2, see especially Corollary 2.

In our applications in section 4, α is determined by the parameters a and b, i. e., the sizes of billiard obstacles. See equation (18). Hence, the results of section 5.2 prove the claims of section 4.2 for generic obstacles. However, (wdd) may fail for some parameters a, b. Our other approach is geared specifically to the cocycles (Ψ_n). We establish their ergodicity for all irrational α in section 5.3.

5.1. **The inequality of Denjoy-Koksma.**

We recall basic facts about continued fractions. See, for instance, [21] for this material. Let $\alpha \in]0, 1[$ be an irrational number, let $[0; a_1, ..., a_n, ...]$ be its continued fraction representation, and let $(p_n/q_n)_{n\geq 0}$ be the sequence of its convergents. The integers p_n (resp. q_n) are the *numerators* (resp. *denominators*) of α. Thus $p_{-1} = 1$, $p_0 = 0$, $q_{-1} = 0$, $q_0 = 1$. For

[10] In our applications $G \simeq \mathbb{R}^{d_1} \times \mathbb{Z}^{d_2}$.

[11] We will use the abbreviation (wdd).

$n \geq 1$ we have

(30) $p_n = a_n p_{n-1} + p_{n-2}, q_n = a_n q_{n-1} + q_{n-2}, (-1)^n = p_{n-1}q_n - p_n q_{n-1}$.

For $u \in \mathbb{R}$ set $\|u\| = \inf_{n\in\mathbb{Z}} |u-n|$. Then for $n \geq 0$ we have $\|q_n\alpha\| = (-1)^n(q_n\alpha - p_n)$. We also have

$$1 = q_n\|q_{n+1}\alpha\| + q_{n+1}\|q_n\alpha\|, \tag{31}$$

$$\frac{1}{q_{n+1}+q_n} \leq \|q_n\alpha\| \leq \frac{1}{q_{n+1}} = \frac{1}{a_{n+1}q_n + q_{n-1}}, \tag{32}$$

$$\|q_n\alpha\| \leq \|k\alpha\| \text{ for } 1 \leq k < q_{n+1}. \tag{33}$$

We denote by $V(\varphi)$ the variation of functions; we will use the shorthand BV for functions of bounded variation. A function is *centered* if $\int_X \varphi(x)\,dx = 0$. Let φ be a centered BV function on X. Let p/q be a rational number in lowest terms such that $\|\alpha - p/q\| < 1/q^2$. The *Denjoy-Koksma inequality* says that for any $x \in X$ we have

$$|\sum_{\ell=0}^{q-1} \varphi(x+\ell\alpha)| \leq V(\varphi). \tag{34}$$

The following is immediate from equations (32) and (34).

Corollary 1. *Let $\Phi : X \to G$ be a centered BV function. Then the cocycle (Φ_n) over any irrational rotation is recurrent.*

We will use the following properties of the sequence p_k/q_k. At least one of any two consecutive numerators (resp. denominators) is odd. If both p_n, q_n are odd, then one of p_{n+1}, q_{n+1} is even.

Lemma 9. *Let α and p_k, q_k for $k \geq 0$ be as above. Then the following holds.* 1. *In any pair of consecutive denominators at least one satisfies $q_n\|q_n\alpha\| < 1/2$.* 2. *Out of any four consecutive denominators at least one is odd and satisfies $q_n\|q_n\alpha\| < 1/2$.*

Proof. 1. For any $n \in \mathbb{N}$ define δ_1 and δ_2 by $q_n\|q_n\alpha\| = \frac{1}{2} - \delta_1$, $q_{n+1}\|q_{n+1}\alpha\| = \frac{1}{2} - \delta_2$. By equation (31), we have

$$(q_{n+1} - q_n)^2 = 2\delta_1 q_{n+1}^2 + 2\delta_2 q_n^2.$$

Hence, δ_1 and δ_2 cannot be both negative.
2. The a priori possible parities for any four consecutive denominators $q_{n-1}, q_n, q_{n+1}, q_{n+2}$ are as follows:

$$(0,\ 1,\ 0,\ 1)\ \ (0,\ 1,\ 1,\ 0)\ \ (1,\ 0,\ 1,\ 0)\ \ (0,\ 1,\ 1,\ 1)$$
$$(1,\ 0,\ 1,\ 1)\ \ (1,\ 1,\ 0,\ 1)\ \ (1,\ 1,\ 1,\ 0)\ \ (1,\ 1,\ 1,\ 1).$$

If there are two consecutive odd denominators, then the statement follows from claim 1. It remains to consider the possibilities $(0,\ 1,\ 0,\ 1)$

and $(1,\ 0,\ 1,\ 0)$. Then we have, respectively, q_n is odd, $a_{n+1} \neq 1$, and q_{n+1} is odd, $a_{n+2} \neq 1$. Set $q = q_n$ (resp. $q = q_{n+1}$) in the former (resp. latter) case. Then $q\|q\alpha\| < 1/2$. ■

5.2. **Ergodicity of generic cocycles.**
Let $d(\cdot, \cdot)$ be an invariant distance on G.

Definition 5. Let $a \in G$. 1. Suppose that for $n \geq 1$ there exist $\ell_n \in \mathbb{N}$, $\varepsilon_n > 0$, and $\delta > 0$ such that

i) We have $\lim_n \varepsilon_n = 0$, $\lim_n \ell_n\alpha \text{ mod } 1\ = 0$,
ii) We have $\text{Leb}(\{x : d(\Phi_{\ell_n}(x), a) < \varepsilon_n\}) \geq \delta$.

Then we say that a is a *quasi-period* for the cocycle (Φ_n).
2. We say that a is a *period* if for every $\rho_{\alpha,\Phi}$-invariant measurable function f on $X \times G$ and for a.e. $(x, g) \in X \times G$ we have

$$f(x, g + a) = f(x, g). \tag{35}$$

We will use the following fact [6].

Lemma 10. *Every quasi-period is a period.*

The set of periods is a closed subgroup of G which coincides with the group of finite essential values of the cocycle [22]. A cocycle is ergodic iff its group of periods is G.

We introduce more notation. Let $\Phi : X \to G$ be a non constant piecewise constant function. Denote by $R(\Phi) \subset G$ the range of Φ, i. e., $a \in R(\Phi)$ iff $\Phi(x) = a$ on a nontrivial interval. Denote by $\mathcal{D} = \{t_i : i = 1, ..., d\}$ the set of discontinuities of Φ. We assume without loss of generality that $0 \in \mathcal{D}$. For $N \in \mathbb{N}$ let $\mathcal{D}_N = \{t_i - j\alpha \mod 1 : 1 \leq i \leq d, 0 \leq j < N\}$ be the set of discontinuities for $\Phi_N(t) = \sum_{k=0}^{N-1} \Phi(t + k\alpha)$. We set $\mathcal{D}_N = \{0 = \gamma_{N,1} < ... < \gamma_{N,dN} < 1\}$; thus, for $1 \leq \ell \leq dN$ the elements $\gamma_{N,\ell}$ run through $\mathcal{D}_N$ in the natural order. We set $\gamma_{N,dN+1} = \gamma_{N,1}$. The following notions will be important in what follows.

Definition 6. 1. Let $0 < \alpha < 1$ be irrational. Let $\Phi : X \to G$ be a piecewise constant function; let (Φ_n) be the corresponding cocycle over ρ_α. Suppose that there is $c > 0$ and an infinite set W of denominators of α such that for all $q \in W, \ell \in \{1, \ldots, dq\}$ we have

$$\gamma_{q,\ell+1} - \gamma_{q,\ell} \geq \frac{c}{q}. \tag{36}$$

Then the cocycle has *well distributed discontinuities*. We will use the shorthand *(wdd)*.

2. Let $\alpha \in]0,1[$ be irrational, and let $[0; a_1, ..., a_n, ...]$ be its continued fraction. Then α has *property (D)* if there is $M \in \mathbb{N}$ such that for infinitely many n either $a_n \in [2, M]$ or $a_n = a_{n+1} = 1$.

Lemma 11. *Let $t \in \frac{1}{2}(\mathbb{Z}\alpha + \mathbb{Z}) \setminus (\mathbb{Z}\alpha + \mathbb{Z})$. Then there exist $c > 0$ and $L \in \mathbb{N}$ such that, if $n \geq L$ and either i) $a_{n+1} \in [2, M]$ or ii) $a_{n+1} = a_{n+2} = 1$, then for $1 \leq k \leq q_n - 1$ we have*

$$\|k\alpha - t\| \geq \frac{c}{q_n}. \tag{37}$$

Proof. Let $t = \frac{1}{2}\ell\alpha + \frac{1}{2}r$, with $\ell, r \in \mathbb{Z}$, and ℓ or r odd. Let L be such that $|\ell| < q_{n-1}$ for $n \geq L$. Let $k \in [1, q_n - 1]$. For $n \geq L$, we have if $a_{n+1} \geq 2$,

$$|2k - \ell| \leq 2k + |\ell| < 2q_n + q_{n-1} \leq a_{n+1}q_n + q_{n-1} = q_{n+1}.$$

If $a_{n+1} \in [2, M]$, then, by equations (32) and (33), for all $j \in [1, q_{n+1}[$, we have

$$\|j\alpha\| \geq \|q_n\alpha\| \geq \frac{1}{q_n + q_{n+1}} = \frac{1}{(a_{n+1} + 1)q_n + q_{n-1}} \geq \frac{1}{(2 + M)q_n}. \tag{38}$$

If $2k - \ell \neq 0$, equation (38) implies

$$\|k\alpha - t\| \geq \frac{1}{2}\|(2k - \ell)\alpha\| \geq \frac{1}{2q_n(2 + M)}.$$

If $a_{n+1} = a_{n+2} = 1$, we have $q_{n+1} = q_n + q_{n-1}$, $q_{n+2} = q_{n+1} + q_n = 2q_n + q_{n-1}$ and $|2k - \ell| < 2q_n + q_{n-1} = q_{n+2}$. Thus, if $2k - \ell \neq 0$

$$\|(2k - \ell)\alpha\| \geq \|q_{n+1}\alpha\| \geq \frac{1}{q_{n+2} + q_{n+1}} = \frac{1}{3q_n + 2q_{n-1}} \geq \frac{1}{5q_n}.$$

If ℓ is even then r is odd, so that for $2k = \ell$ we have $\|k\alpha - t\| = \|(k - \frac{1}{2}\ell)\alpha - \frac{1}{2}\| = \frac{1}{2}$.

In each case equation (37) holds with $c = \inf(\frac{1}{2(2+M)}, \frac{1}{10})$. ∎

The following is immediate from Lemma 11.

Proposition 11. *Let α be irrational, let $\Phi : X \to G$ be a piecewise constant function, and let $\mathcal{D} \subset X$ be its set of discontinuities. If α has property (D) and if $\mathcal{D} \subset \{\frac{1}{2}(\mathbb{Z}\alpha + \mathbb{Z}) \setminus (\mathbb{Z}\alpha + \mathbb{Z})\} \mod 1$, then the cocycle (Φ_n) has (wdd).*

The following theorem is the main result of this subsection.

Theorem 7. *Let $0 < \alpha < 1$ be irrational. Let $\Phi : X \to \mathbb{Z}^r$ be a piecewise constant, centered function, and let $R(\Phi) \subset \mathbb{Z}^r$ be the range of Φ.*

Let (Φ_n) be the corresponding cocycle. If (Φ_n) has property (wdd), then the group of its periods contains $R(\Phi)$.

Proof. Let $W \subset \mathbb{N}$ be the infinite set of denominators of α introduced in Definition 6. We will study the family of functions $\{\Phi_q(x) = \sum_{0\le k\le q-1} \Phi(x + k\alpha \mod 1) : q \in W\}$. For $1 \le \ell \le dq$ let $I_{q,\ell} =]\gamma_{q,\ell}, \gamma_{q,\ell+1}[$ be the intervals of continuity. Set $\mathcal{I}_q = \{I_{q,\ell} : 1 \le \ell \le dq\}$.

For t_i in the set $\mathcal{D}$ of discontinuities of Φ, let $\sigma_i = \lim_{\varepsilon\to 0^+}[\Phi(t_i + \varepsilon) - \Phi(t_i - \varepsilon)]$ be the jump at t_i; let $\Sigma(\Phi) = \{\sigma_i : 1 \le i \le d\}$ be the set of jumps. Set $R = \cup_{q\in W} R(\Phi_q) \subset \mathbb{Z}^r$. By equation (34), R is a finite set.

Each interval $[\frac{k}{q}, \frac{k+1}{q}[$, $0 \le k < q$, contains an element $j\alpha \mod 1$, $0 \le j \le q-1$. Thus, for any $t \in X$, the elements $\{t + j\alpha \mod 1 : j = 0, \dots, q-1\}$ partition X into intervals of lengths less than $2/q$. Hence, any interval $J \subset X$ of length $\ge 2/q$ contains at least one point of the set $\{t + j\alpha \mod 1 : j = 0, \dots, q-1\}$.

Let c be the constant from equation (36); set $L = \lfloor \frac{2}{c} \rfloor + 1$. Let $I_{q,\ell} \in \mathcal{I}_q$ be arbitrary. Set $J_{q,\ell} \subset X$ be the union of L consecutive intervals in $\mathcal{I}_q$ starting with $I_{q,\ell}$. By equation (36), the length of $J_{q,\ell}$ is greater than or equal to $2/q$. Thus, for any $t_i \in \mathcal{D}$, the interval $J_{q,\ell}$ contains a point in the set $\{t_i + j\alpha \mod 1 : j = 0, \dots, q-1\}$. Therefore, for any $\sigma \in \Sigma(\Phi)$, there is $v \in R$ and two consecutive intervals $I, I' \in \mathcal{I}_q$ such that $I \cup I' \subset J_{q,\ell}$ and such that Φ_q takes values v and $v + \sigma$ on I and I' respectively.

Let $\sigma \in \Sigma(\Phi)$, $v \in R$. Let $\mathcal{F}_q(\sigma) \subset \mathcal{I}_q$ be the family of intervals $I \in \mathcal{I}_q$ such that the jump at the right endpoint of I is σ. Let $\mathcal{A}_q(\sigma, v) \subset \mathcal{F}_q(\sigma)$ be the set of intervals $I \in \mathcal{F}_q(\sigma)$ such that the value of Φ_q on I is v; let $\mathcal{A}'_q(\sigma, v) \subset \mathcal{F}_q(\sigma)$ be the set of intervals $I' \in \mathcal{F}_q(\sigma)$ adjacent on the right to the intervals $I \in \mathcal{A}_q(\sigma, v)$. Let $A_q(\sigma, v) \subset X$ (resp. $A'_q(\sigma, v) \subset X$) be the union of intervals $I \in \mathcal{A}_q(\sigma, v)$ (resp. $I' \in \mathcal{A}'_q(\sigma, v)$). Thus, Φ_q takes value v (resp. $v + \sigma$) on $A_q(\sigma, v)$ (resp. $A'_q(\sigma, v)$).

Denote by $|\cdot|$ the cardinality of a set. There is $v_0 \in R$ and an infinite subset of W (which we denote by W again) such that for $q \in W$ we have

$$|\mathcal{A}_q(\sigma, v_0)|, |\mathcal{A}'_q(\sigma, v_0)| \ge \frac{1}{|R|} |\mathcal{F}_q(\sigma)|. \tag{39}$$

We have $|\mathcal{F}_q(\sigma)| \geq qd/L$. By equation (36) and equation (39)

$$\operatorname{Leb}\left(A_q(\sigma, v_0)\right), \operatorname{Leb}\left(A_q'(\sigma, v_0)\right) \geq \frac{1}{|R|}\frac{qd}{L}\frac{c}{q} \geq \frac{dc^2}{(2+c)|R|}.$$

Thus, both v_0 and $v_0+\sigma$ are quasi-periods for the cocycle (Φ_n). By Lemma 10, they are periods. Hence σ is a period for the cocycle (Φ_n). Since $\sigma \in \Sigma(\Phi)$ was arbitrary, the group of periods for $\rho_{\alpha,\Phi}$ contains $\Sigma(\Phi)$.

If $H \subset G$ is a subgroup, we will denote by "bar" the reduction modulo H. Thus, $\Phi : X \to G/H$ is a piecewise constant function. The dynamical system $(X \times G/H, \rho_{\alpha,\bar{\Phi}}, \bar{\mu})$ is the skew product over ρ_α with the fiber G/H and the displacement function $\bar{\Phi}$. To simplify notation, we will denote it by $\overline{\rho_{\alpha,\Phi}}$.

Let $H \subset G$ (resp. $H' \subset G$) be the group generated by $\Sigma(\Phi)$ (resp. $R(\Phi)$). We have shown that H is contained in the group of periods for $\rho_{\alpha,\Phi}$. The function $\bar{\Phi} : X \to G/H$ is constant. Let $a \in H'$ be such that $\bar{\Phi} = \bar{a}$. Then

$$\overline{\rho_{\alpha,\Phi}}(x, \bar{g}) = (\rho_\alpha(x), \bar{g} + \bar{a}).$$

Observe that $H'/H \subset G/H$ is the cyclic group generated by $\bar{a}$. If $|H'/H| = \infty$, then $\rho_{\alpha,\bar{\Phi}}$ is dissipative, contrary to Corollary 1. Thus, H'/H is a finite cyclic group. Let $|H'/H| = n$.

Let f be a $\rho_{\alpha,\Phi}$-invariant measurable function. Then f defines a $\overline{\rho_{\alpha,\Phi}}$-invariant function; we denote it by $\bar{f}$. By the above equation

$$\bar{f}(\rho_\alpha^n(x), \bar{g}) = \bar{f}(x, \bar{g}).$$

Hence $\bar{f}(x, \bar{g})$ depends only on $\bar{g}$. In the self-explanatory notation, $\bar{f}(x, \bar{g}) = \bar{f}(\bar{g})$. Therefore, $\bar{a}$ is a period for $\bar{f}$, and hence a is a period for f. ∎

The following is immediate from Proposition 11 and Theorem 7.

Corollary 2. *Let $\Phi : X \to G$ be a piecewise constant, centered function such that $R(\Phi)$ generates G. Let $\mathcal{D} \subset X$ be the set discontinuities of Φ. Let $0 < \alpha < 1$ be irrational. Suppose that $\mathcal{D} \subset \{\frac{1}{2}(\mathbb{Z}\alpha + \mathbb{Z}) \setminus (\mathbb{Z}\alpha + \mathbb{Z})\}$* mod 1. *If α has property (D), then the skew product $\rho_{\alpha,\Phi}$ is ergodic.*

Remark 3. Let $\Psi : X \to \mathbb{Z}^2$ be the piecewise constant function that arose in our analysis of the rectangular Lorenz gas. See equation (21). It satisfies the assumptions of Corollary 2. Almost every irrational α has property (D). Thus, Corollary 2 implies the claim of Theorem 5 for generic small obstacles. The results in the next section will allow us to prove Theorem 5 for arbitrary small obstacles.

5.3. **Removing the genericity assumptions.**
Let $0 < \alpha < 1$. Set

$$\gamma = 1_{[0,\frac{1}{2}[} - 1_{[\frac{1}{2},1[}, \ \zeta = 1_{[0,\frac{1}{2}-\frac{\alpha}{2}[} - 1_{[\frac{1}{2},1-\frac{\alpha}{2}[}. \tag{40}$$

Thus, $\gamma, \zeta : X \to \mathbb{Z}$ are piecewise constant functions. See Figure 10.

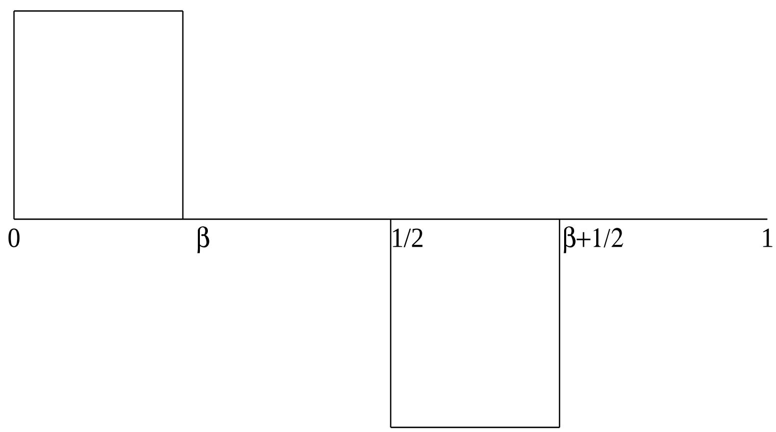

FIGURE 10. The function $\zeta = 1_{[0,\beta[} - 1_{[\frac{1}{2},\beta+\frac{1}{2}[}$ for $\beta < 1/2$.

We will establish the ergodicity of cocycles $(\gamma_n), (\zeta_n)$ for any irrational α. First, we explain the heuristics. Suppose that α does not satisfy property (D). Then, passing to a subsequence, if need be, we have $a_n \to \infty$. Suppose that for all sufficiently large n the numbers q_{2n} and p_{2n} are odd, while q_{2n+1} is even and p_{2n+1} is odd. Then for $n > n_0$ the inequality $\zeta(q_n, \cdot) \neq 0$ holds only on sets of very small measure. Thus, we cannot use the method of Theorem 7. Instead, we will consider $\zeta(tq, \cdot)$ for such t that $||tq\alpha||$ is close to zero but big enough to ensure that $\zeta(tq, x) = \sum_{k=0}^{t-1} \zeta(q, x + kq\alpha)$ does not vanish on a set of measure bounded away from zero.

We will informally refer to this idea as the "filling method". Figure 11 illustrates it.

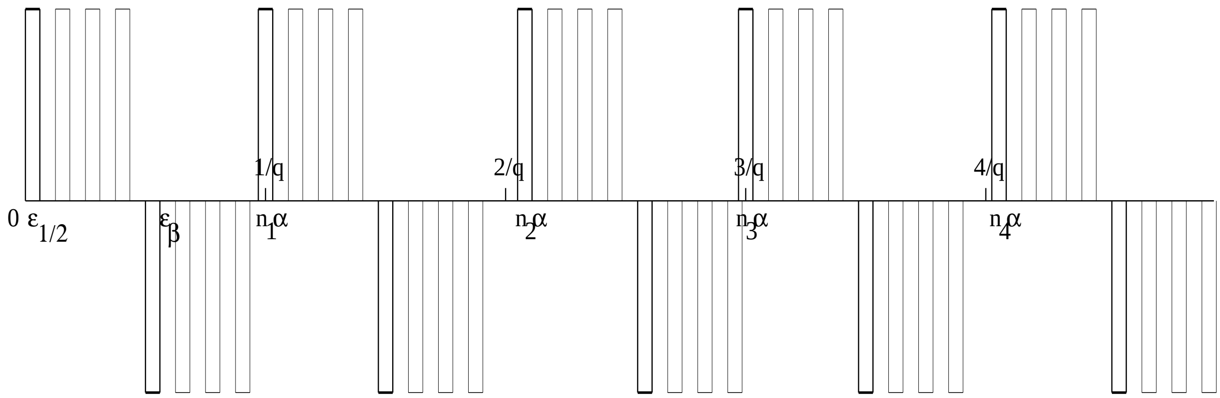

FIGURE 11. *Illustration of the filling method: Graph of $\zeta(t_n q_n, \cdot)$.*

From now on, $0 < \alpha < 1$ is an arbitrary irrational number. We leave the proof of the following lemma to the reader.

Lemma 12. *If q is odd and $q\|q\alpha\| < 1/2$, then for all x we have $\sum_{j=0}^{q-1} \gamma(x + j\alpha) = \pm 1$.*

By Lemma 12, 1 is a quasi-period for the cocycle (γ_n) over ρ_α. By Lemma 10, (γ_n) is ergodic.

Theorem 8. *Let $\zeta : X \to \mathbb{Z}$ be given by equation (40). Then the cocycle (ζ_n) over ρ_α is ergodic.*

Proof. Let p_n/q_n be the convergents of α. Let $p'_n, q'_n \in \mathbb{N}$ be such that $q_n = 2q'_n$ or $q_n = 2q'_n + 1$ and $p_n = 2p'_n$ or $p_n = 2p'_n + 1$, depending on the parity. Set $\alpha = p_n/q_n + \theta_n$.

The set of discontinuities of ζ is $\{0, \beta = \frac{1}{2} - \frac{\alpha}{2}, \frac{1}{2}, \beta' = 1 - \frac{\alpha}{2}\}$; the respective jumps are $1, -1, -1, 1$. If $t \in \{0, \beta, \frac{1}{2}, \beta'\}$, the corresponding discontinuities of ζ_q are $\{t - j\alpha : j = 0, ..., q-1\}$.

Depending on the parities of p_n, q_n, we define partitions $\{0, \beta, \frac{1}{2}, \beta'\} = P_1 \cup P_2$ as follows:

1) For q_n odd and p_n even, we set $P_1 = \{0, \beta'\}$, $P_2 = \{\frac{1}{2}, \beta\}$;
2) For q_n even and p_n odd, we set $P_1 = \{0, \frac{1}{2}\}$, $P_2 = \{\beta, \beta'\}$;
3) For q_n odd and p_n odd, we set $P_1 = \{0, \beta\}$, $P_2 = \{\frac{1}{2}, \beta'\}$.

Discontinuities of ζ_{q_n} which come from points in the same atom of the partition are very close to each other; discontinuities which come from points in distinct atoms of the partition are well separated from each other.

We will consider in detail only case 2). The analysis of other cases is similar, and we leave it to the reader. In what follows, all numbers and equalities are understood $\mod 1$. To simplify notation, we will suppress the subscript n. Thus, we write $\alpha = p/q + \theta$, etc.

a) The set of discontinuities, $D_0 \subset X$, corresponding to $t = 0$ is $D_0 = \{-j\alpha : j = 0, ..., q-1\}$. For each integer r there is $j_1(r) \in \{0, ..., q-1\}$ such that $-j_1(r)p = r \mod q$. Hence, $D_0 = \{\frac{r}{q} - j_1(r)\theta :\ r = 0, ..., q-1\}$.
b) Let $t = \frac{1}{2}$. The corresponding set of discontinuities is $D_{\frac{1}{2}} = \{\frac{(q'-jp)}{q} - j\theta : j = 0, ..., q-1\}$. For each integer r there is $j_2(r) \in \{0, ..., q-1\}$ such that $q' - j_2(r)p = r \mod q$. Thus, $D_{\frac{1}{2}} = \{\frac{r}{q} - j_2(r)\theta,\ r = 0, ..., q-1\}$.
c) Let $t = 1 - \frac{\alpha}{2}$. Then the set of discontinuities is $D_{1-\frac{\alpha}{2}} = \{\frac{1}{2q} - \frac{(p'+1+jp)}{q} - (j + \frac{1}{2})\theta : j = 0, ..., q-1\}$. For each integer r there is $j_3(r) \in \{0, ..., q-1\}$ such that $-(p' + 1 + j_3(r)p) = r \mod q$. Hence $D_{1-\frac{\alpha}{2}} = \{\frac{r}{q} + \frac{1}{2q} - (j_3(r) + \frac{1}{2})\theta :\ r = 0, ..., q-1\}$.
d) Let $t = \frac{1}{2} - \frac{\alpha}{2}$. The set of discontinuities is $D_{\frac{1}{2}-\frac{\alpha}{2}} = \{\frac{1}{2q} - \frac{(-q'+p'+1+jp)}{q} - (j + \frac{1}{2})\theta : j = 0, \dots, q-1\}$. For each integer r there is $j_4(r) \in \{0, ..., q-1\}$ such that $q' - (p' + 1 + j_4(r)p) = r \mod q$. Hence $D_{\frac{1}{2}-\frac{\alpha}{2}} = \{\frac{r}{q} + \frac{1}{2q} - (j_4(r) + \frac{1}{2})\theta :\ r = 0, ..., q-1\}$.

The set of discontinuities of ζ_q is $D_0 \cup D_{\frac{1}{2}} \cup D_{1-\frac{\alpha}{2}} \cup D_{\frac{1}{2}-\frac{\alpha}{2}}$. Observe that in all cases $|j_i(r)\theta| \le |q\theta|$; since $(j_2 - j_1)p = \frac{1}{2}q \mod q$ and $(j_4 - j_3)p = \frac{1}{2}q \mod q$, we have

$$j_2(r) = j_1(r) \pm \frac{1}{2}q,\ j_4(r) = j_3(r) \pm \frac{1}{2}q. \tag{41}$$

We are going to determine the values taken by the cocycle $\zeta_q(x)$ for x in a neighborhood of the typical interval $[\frac{r}{q}, \frac{r+1}{q}]$, where r is an integer in $\{1, ..., q-1\}$. Assume, for concreteness, that $\theta < 0$, $j_1 = j_2 + \frac{1}{2}q$, $j_4 = j_3 + \frac{1}{2}q$.[12] Let x start at $\frac{r}{q}$ and let it move to the right; set $\zeta_q(x) = a$. The value of the cocycle $\zeta_q(x)$ is constant until x crosses the discontinuity (corresponding to $t = 0$) at $\frac{r}{q} - j_1(r)\theta$, where $\zeta_q(x)$ increases by 1. After that the cocycle does not change until x crosses the discontinuity at $\frac{r}{q} - j_2(r)\theta$ (corresponding to $t = \frac{1}{2}$) where the cocycle decreases by 1, returning to the value a.

[12] The analysis for $\theta > 0$ and/or $j_1 = j_2 - \frac{1}{2}q$, and/or $j_4 = j_3 - \frac{1}{2}q$ is analogous.

The first two discontinuities occur before x crosses the two other discontinuities under the condition that $|j_i(r)\theta|$ is less than $\frac{1}{2q}$. This takes place if $q^2|\theta| < \frac{1}{2}$, a condition which holds below because we consider the case when $q^2|\theta|$ is small. As x continues to move to the right, the cocycle remains at the value a until, near $\frac{r}{q} + \frac{1}{2q}$, it increases by 1 at the point $\frac{r}{q} + \frac{1}{2q} - (j_3(r) + \frac{1}{2})\theta$, a discontinuity corresponding to $t = 1 - \frac{\alpha}{2}$, and then decreases by 1 at $\frac{r}{q} + \frac{1}{2q} - (j_4(r) + \frac{1}{2})\theta$, a discontinuity corresponding to $t = \frac{1}{2} - \frac{\alpha}{2}$.

Therefore, we have

$$\begin{aligned}
\zeta_q &= a \pm 1 \text{ on }]\frac{r}{q} - j_1(r)\theta, \frac{r}{q} - j_2(r)\theta[, \\
\zeta_q &= a \pm 1 \text{ on }]\frac{r}{q} + \frac{1}{2q} - (j_3(r) + \frac{1}{2})\theta, \frac{r}{q} + \frac{1}{2q} - (j_4(r) + \frac{1}{2})\theta[.
\end{aligned}$$

Elsewhere, $\zeta_q = a$.

This analysis is valid for every interval $[\frac{r}{q}, \frac{r+1}{q}]$. The order of the discontinuity points may change, but not the order between the groups of discontinuity. In particular, this implies that $\zeta_q = a$ on a subset of large measure in $[0, 1]$. Since the mean value of ζ_q is zero, we have $a = 0$.

We will now finish the proof. If α satisfies condition (D), then the claim holds, by Corollary 2. Thus, we assume that α does not satisfy condition (D). Then one of the cases 1), 2), 3) materializes for an infinite sequence (n_k) such that $a_{n_k+1} \to \infty$. We will then say, for brevity, that a case *occurs infinitely often.* If case 1) occurs infinitely often, then 1 is a quasi-period for (ζ_n). See figure 12.

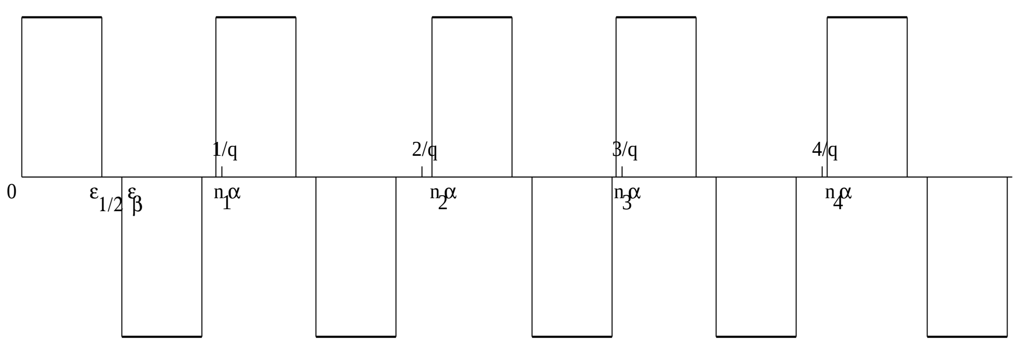

FIGURE 12. *Graph of the cocycle ζ_q for q odd, p even. Its value is* 1 *on a set of measure* $\geq \delta > 0$.

Suppose now that case 2) occurs infinitely often. We will use the preceding analysis. For $1 \leq r \leq q_{n_k} - 1$ set

$$\begin{aligned} I_{k,r} &=]\frac{r}{q_{n_k}} - j_1(r)\theta_{n_k}, \frac{r}{q_{n_k}} - j_2(r)\theta_{n_k}[, \\ J_{k,r} &=]\frac{r}{q_{n_k}} + \frac{1}{2q_{n_k}} - (j_3(r) + \frac{1}{2})\theta_{n_k}, \frac{r}{q_{n_k}} + \frac{1}{2q_{n_k}} - (j_4(r) + \frac{1}{2})\theta_{n_k}[. \end{aligned}$$

By equation (41), these intervals have length $\frac{1}{2}q_{n_k}|\theta_{n_k}|$. At the scale $\frac{1}{q_{n_k}}$, they are close to $\frac{r}{q_{n_k}}$ and to $\frac{r}{q_{n_k}} + \frac{1}{2q_{n_k}}$ respectively. Outside of these intervals, $\zeta(q_{n_k}, \cdot) = 0$.

Let $\delta \in]0, \frac{1}{4}[$. Set $t_k = \lfloor \delta a_{n_k+1} \rfloor$. For $J \subset X$ and $u \in \mathbb{R}$, we set $(J + u) = J + u \mod 1$. Set

$$A_k = \cup_{j=0}^{q_{n_k}-1} \cup_{s=0}^{t_k-1} (I_{k,j} - sq_{n_k}\alpha), \; B_k = \cup_{j=0}^{q_{n_k}-1} \cup_{s=0}^{t_k-1} (J_{k,j} - sq_{n_k}\alpha).$$

The distance between the intervals $I_{k,r}$ and $J_{k,r}$ is at least $\frac{1}{2q_{n_k}} - q_{n_k}|\theta_{n_k}|$. Since, by the choice of t_k, we have $q_{n_k}|\theta_{n_k}|t_k \leq \frac{1}{2q_{n_k}} - q_{n_k}|\theta_{n_k}|$, the translated intervals in the definition of A_k and B_k do not overlap.

Let us consider the cocycle at time $t_k q_{n_k}$. We have $\zeta(t_k q_{n_k}, x) = \sum_{s=0}^{t_k-1} \zeta(q_{n_k}, x + sq_{n_k}\alpha)$. By the preceding analysis of the values of ζ_q, we have $\zeta(t_k q_{n_k}, \cdot) = \pm 1$ on A_k and B_k. Also

$$\text{Leb}(A_k) = \frac{1}{2}t_k q_{n_k} q_{n_k}|\theta_{n_k}| \geq \frac{1}{2}\delta a_{n_k+1}\frac{q_{n_k}}{q_{n_k+1}} \geq \frac{1}{2}\delta > 0.$$

Since $t_k q_{n_k}\alpha \mod 1 \to 0$, and since on A_k the cocycle takes at most the two values ± 1, we have shown that 1 or -1 is a quasi-period for the cocycle (ζ_n).

The possibility that case 3) occurs infinitely often is analyzed the same way, with the conclusion that 1 is a quasi-period.

Thus, no matter which of the three cases occurs infinitely often, 1 is a quasi-period for the cocycle (ζ_n). The claim now follows, by Lemma 10. ∎

We will now establish the main result of this subsection.

Theorem 9. *Let $\Psi : X \to \mathbb{Z}^2$ be the function defined by equation* (21). *Then, over any irrational rotation $\rho_\alpha : x \to x + \alpha \mod 1$ on the circle, the corresponding cocycle is ergodic.*

Proof. Set $\Psi = (\psi_1, \psi_2)$ and $\beta = \frac{1}{2} - \frac{\alpha}{2}$. The functions ψ_1, ψ_2 satisfy

$$\psi_1(x) + \psi_2(x) = \gamma(x), \; \psi_1(x) - \psi_2(x) = \gamma(x + \beta).$$

By Lemma 9 and Lemma 12, for an infinite sequence (n_k) we have $\gamma(q_{n_k}, x), \gamma(q_{n_k}, x+\beta) \in \{\pm 1\}$. This implies the existence of measurable sets $A_k \subset X$ satisfying $\mathrm{Leb}(A_k) > \frac{1}{4}$, and such that for $x \in A_k$ the vector function $(\gamma(q_{n_k}, x), \gamma(q_{n_k}, x+\beta))$ is constant. Its values are $(+1, +1)$, $(+1, -1)$, $(-1, +1)$, or $(-1, -1)$.

Thus, for $x \in A_k$, the vector function $(\psi_1(q_{n_k}, x), \psi_2(q_{n_k}, x))$ is identically $(1, 0)$, $(0, 1)$, $(-1, 0)$, or $(0, -1)$. Hence, one of the elements $(\pm 1, 0), (0, \pm 1) \in \mathbb{Z}^2$ is a quasi-period for the cocycle (Ψ_n). Suppose, for instance, that $(1, 0)$ is a quasi-period. Hence, by Lemma 10, $(1, 0)$ is a period. Let f be a $\rho_{\alpha,\Psi}$-invariant function. It defines a $\rho_{\alpha,\zeta}$-invariant function on $X \times \mathbb{Z}$. Here $\zeta : X \to \mathbb{Z}$ is given by equation (40). By Theorem 8, $f = \mathrm{const}$, i. e., the cocycle (Ψ_n) is ergodic. The other cases are disposed of the same way. ∎

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References

[1] J. Aaronson, *An introduction to infinite ergodic theory*, AMS, Providence, R.I, 1997.

[2] G. Atkinson, *Recurrence of co-cycles and random walks*, J. London Math. Soc. **13** (1976), 486 – 488.

[3] P. Bachurin, K. Khanin, J. Marklof, and A. Plakhov, *Perfect retroreflectors and billiard dynamics*, preprint arXiv:0911.1984 (2009).

[4] N. Chevallier and J.-P. Conze, *Examples of recurrent or transient stationary walks in $\mathbb{R}^d$ over a rotation of $\mathbb{T}^2$*, pp. 71 – 84 in *Contemporary Mathematics* **485**, AMS, Providence, R.I, 2009.

[5] J.-P. Conze, *Transformations cylindriques et mesures finies invariantes*, Ann. Sci. Univ. Clermont Math. **17** (1979), 25 – 31.

[6] J.-P. Conze, *Recurrence, ergodicity and invariant measures for cocycles over a rotation*, pp. 45 – 70 in *Contemporary Mathematics* **485**, AMS, Providence, R.I, 2009.

[7] I.P. Cornfeld, S.V. Fomin, Ya.G. Sinai, *Ergodic theory*, Springer-Verlag, New York, 1982.

[8] G. Cristadoro, M. Lenci, M. Seri, *Recurrence for quenched random Lorentz tubes*, preprint arXiv:0909.3069 (2009).

[9] P. Ehrenfest and T. Ehrenfest, *Encyclopedia article (1912)*; english translation in *"The conceptual foundations of the statistical approach in mechanics"*, Cornell University Press, Ithaca, N.Y., 1959.

[10] E. Gutkin, *Billiard flows on almost integrable polyhedral surfaces*, Erg. Theory & Dyn. Syst. **4** (1984), 560 – 584.

[11] E. Gutkin, *Billiards in polygons: Survey of recent results*, J. Stat. Phys. **83** (1996), 7 – 26.
[12] E. Gutkin, *Billiard dynamics: A survey with the emphasis on open problems*, Reg. & Chaot. Dyn. **8** (2003), 1 – 13.
[13] E. Gutkin, *Geometry, topology and dynamics of geodesic flows on noncompact polygonal surfaces*, Reg. & Chaot. Dyn. **15** (2010), 482 – 503.
[14] E. Gutkin and C. Judge, *Affine mappings of translation surfaces: geometry and arithmetic*, Duke Math. J. **103** (2000), 191 – 213.
[15] E. Gutkin and A. Katok, *Caustics for inner and outer billiards*, Comm. Math. Phys. **173** (1995), 101 – 133.
[16] J. Hardy and J. Weber, *Diffusion in a periodic wind-tree model*, J. Math. Phys. **21** (1980), 1802 – 1808.
[17] W.P. Hooper, B. Weiss, *Generalized staircases: recurrence and symmetry*, preprint arXiv:0905.3736 (2009).
[18] P. Hubert, B. Weiss, *Ergodicity for infinite periodic translation surfaces*, preprint (2010).
[19] S. Kerckhoff, H. Masur, J. Smillie, *Ergodicity of billiard flows and quadratic differentials*, Ann. Math. **124** (1986), 293 – 311.
[20] H. Kesten, *Sums of stationary sequences cannot grow slower than linearly*, Proc. Amer. Math. Soc. **49** (1975), 205 – 211.
[21] A.Ya. Khinchin, *Continued fractions*, Dover Publications, Mineola, N.Y, 1997.
[22] K. Schmidt, *Lectures on Cocycles of Ergodic Transformations Groups*, Lect. Notes in Math. **1**, MacMillan Co. of India, 1977.
[23] Ya. Vorobets, *Ergodicity of billiards in polygons*, Sb. Math. **188** (1997), 389 – 434.

IRMAR, CNRS UMR 6625, Université de Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France
E-mail address: `conze@univ-rennes1.fr`

Copernicus University, Chopina 12/18, Torun 87-100; IMPAN, Sniadeckich 8, Warszawa 10, Poland
E-mail address: `gutkin@mat.umk.pl,gutkin@impan.pl`